Spin in an arbitrary gravitational field

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Abstract

We study the quantum mechanics of a Dirac fermion on a curved spacetime manifold. The metric of the spacetime is completely arbitrary, allowing for the discussion of all possible inertial and gravitational field configurations. In this framework, we find the Hermitian Dirac Hamiltonian for an arbitrary classical external field (including the gravitational and electromagnetic ones). In order to discuss the physical content of the quantum-mechanical model, we further apply the Foldy-Wouthuysen transformation, and derive the quantum equations of motion for the spin and position operators. We analyse the semiclassical limit of these equations and compare the results with the dynamics of a classical particle with spin in the framework of the standard Mathisson-Papapetrou theory and in the classical canonical theory. The comparison of the quantum mechanical and classical equations of motion of a spinning particle in an arbitrary gravitational field shows their complete agreement.

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I. INTRODUCTION

Immediately after the notion of spin was introduced in physics and after the relativistic Dirac theory was formulated, the study of spin dynamics in a curved spacetime (i.e., in the gravitational field) was initiated. The early efforts were mainly concerned with the development of mathematical tools and methods appropriate for the description of the interaction of spinning particles with the gravitational field [1–11]. As an interesting by-product, the studies of the spinor analysis in the framework of the general Lagrange-Noether approach have subsequently resulted in the construction of the gauge-theoretic models of physical interactions, including also gravity [12–16].

At a later stage, considerable attention was turned to the investigation of the specific physical effects in the gravitational field predicted for quantum, semiclassical, and classical relativistic particles with spin [17–32]. Various aspects of the dynamics of fermions were studied in the weak gravitational field, i.e., for the case when the geometry of the spacetime does not significantly deviate from the flat Minkowski manifold. Another class of problems was the analysis of trajectories of semiclassical and classical particles in the gravitational field configurations which arise as the exact solutions of Einstein's equations (such as the spherically symmetric Schwarzschild metric or the Kerr metric of a rotating source). The behavior of spin in the strong gravitational fields represents another interesting subject with the possible applications to the study of the physical processes in the vicinity of massive astrophysical objects and near black holes. For the overview of the important mathematical subtleties, the reader can consult [33–36], for example.

In this paper, we continue our investigations of the quantum and semiclassical Dirac fermions using the method of the Foldy-Wouthuysen (FW) transformation. Earlier, we analyzed the dynamics of spin in weak static and stationary gravitational fields [37–40] and in strong stationary gravitational fields [41] of massive compact sources. Now we extend our previous results to the general case of a completely arbitrary gravitational field.

The paper is organized as follows. In Sec. II, we give preliminaries for the description of the general metric and the coframe, and then derive the Hermitian Dirac Hamiltonian in an arbitrary curved spacetime. For completeness, we consider the electrically charged particle interacting also with the electromagnetic field. In Sec. III, we outline the FW technique and apply this method to derive the FW Hamiltonian together with the corresponding operator equations of motion. The central result is the derivation of the precession of spin in an arbitrary gravitational field. The quantum and semiclassical spin dynamics is compared with the dynamics of a classical spin in Sec. IV. We use the standard formalism of Mathisson and Papapetrou, and discuss the Hamiltonian approach. The results obtained are summarized in Sec. V.

Our notations and conventions are the same as in [38]. In particular, the world indices of the tensorial objects are denoted by Latin letters $i, j, k, \ldots = 0, 1, 2, 3$ and the first letters of the Greek alphabet label the tetrad indices, $\alpha, \beta, \ldots = 0, 1, 2, 3$. Spatial indices of 3-dimensional objects are denoted by Latin letters from the beginning of the alphabet, $a, b, c, \ldots = 1, 2, 3$. The particular values of tetrad indices are marked by hats.

II. DIRAC PARTICLE IN A GRAVITATIONAL FIELD

A. General parametrization of the spacetime metric

Let us recall some basic facts and introduce the notions and objects related to the description of the motion of a classical spinning particle in a curved manifold. The massive particle is quite generally characterized by its position in spacetime, $x^i(\tau)$, where the local spacetime coordinates are considered as functions of the proper time τ, and by the tensor of spin $S^{\alpha\beta} = -\, S^{\beta\alpha}$. The analysis of the dynamics of the classical spinning particle is given later in Sec. IV.

We denote 4-velocity of a particle $U^\alpha = e_i^\alpha dx^i/d\tau$. In view of the choice of parametrization by the proper time, it is normalized by the standard condition $g_{\alpha\beta}U^\alpha U^\beta = c^2$ where $g_{\alpha\beta} = \mathrm{diag}(c^2, -1, -1, -1)$ is the flat Minkowski metric. We use the tetrad e_i^α (or coframe) to describe the dynamics of spinning particles on a spacetime manifold in arbitrary curvilinear coordinates. When the spacetime is flat, which means that the gravitational field is absent, one can choose the global Cartesian coordinates and the holonomic orthonormal frame that coincides with the natural one, $e_i^\alpha = \delta_i^\alpha$. The spacetime metric is related to the coframe field in the usual way: $g_{\alpha\beta} e_i^\alpha e_i^\beta = g_{ij}$.

We use the notation t for the coordinate time and x^a $(a = 1, 2, 3)$ denote spatial local coordinates. There are many different ways to represent a general spacetime metric. A convenient parametrization of the spacetime metric was proposed by De Witt [42] in the context of the canonical formulation of the quantum gravity theory. In a slightly different disguise, the general form of the line element of an arbitrary gravitational field reads

$$ds^2 = V^2 c^2 dt^2 - \delta_{\widehat{a}\widehat{b}} W^{\widehat{a}}{}_c W^{\widehat{b}}{}_d \,(dx^c - K^c cdt)\,(dx^d - K^d cdt). \tag{2.1}$$

This parametrization involves more functions than the actual number of the metric components. Indeed, the total number of the functions $V(t, x^c)$, $K^a(t, x^c)$, and $W^{\widehat{a}}{}_b(t, x^c)$ is $1 + 3 + 9 = 13$ which is greater than 10. However, we have to take into account that the line element (2.1) is invariant under redefinitions $W^{\widehat{a}}{}_b \longrightarrow L^{\widehat{a}}{}_{\widehat{c}} W^{\widehat{c}}{}_b$ using arbitrary local rotations $L^{\widehat{a}}{}_{\widehat{c}}(t, x) \in SO(3)$. Subtracting the 3 rotation degrees of freedom, we recover exactly 10 independent variables that describe the general metric of the spacetime.

B. Dirac equation

One needs the orthonormal frames to discuss the spinor field and to formulate the Dirac equation. From the mathematical point of view, the tetrad is necessary to "attach" a spinor space at every point of the spacetime manifold. Tetrads (coframes) are naturally defined up to a local Lorentz transformations, and one usually fixes this freedom by choosing a gauge. We discussed the choice of the tetrad gauge in [40] and have demonstrated that a physically preferable option is the Schwinger gauge [43, 44], namely the condition $e_a^{\widehat{0}} = 0, a = 1, 2, 3$. Accordingly, for the general metric (2.1) we will work with the tetrad

$$e_i^{\widehat{0}} = V\,\delta^0_i, \qquad e_i^{\widehat{a}} = W^{\widehat{a}}{}_b \left(\delta^b_i - cK^b\,\delta^0_i\right), \qquad a = 1, 2, 3. \tag{2.2}$$

The inverse tetrad, such that $e^i_\alpha e^\alpha_j = \delta^i_j$,

$$e^i_{\widehat{0}} = \frac{1}{V}\left(\delta^i_0 + \delta^i_a cK^a\right), \qquad e^i_{\widehat{a}} = \delta^i_b W^b{}_{\widehat{a}}, \qquad a = 1, 2, 3, \tag{2.3}$$

also satisfies the similar Schwinger condition, $e^0_{\widehat{a}} = 0$. Here we introduced the inverse 3×3 matrix, $W^a{}_{\widehat{c}} W^{\widehat{c}}{}_b = \delta^a_b$.

The following observation will be useful for the subsequent computations. A classical massive particle moves along a world line $x^i(\tau), i = 0, 1, 2, 3$, parametrized by the proper time τ. Its 4-velocity is defined as usual by the derivatives $U^i = dx/d\tau$. With respect to a given orthonormal frame, the velocity has the components $U^\alpha = e^\alpha_i U^i, \alpha = 0, 1, 2, 3$. It is convenient to describe the 4-velocity by its 3 spatial components $v^{\widehat{a}}, a = 1, 2, 3$, in an anholonomic frame. Then $U^\alpha = (\gamma, \gamma v^{\widehat{a}})$, with the Lorentz factor $\gamma^{-1} = \sqrt{1 - v^2/c^2}$, and, consequently,

$$U^0 = \frac{dt}{d\tau} = e^0_\alpha U^\alpha = \frac{\gamma}{V}, \tag{2.4}$$

$$U^a = \frac{dx^a}{d\tau} = e^a_\alpha U^\alpha = \frac{\gamma}{V}\left(cK^a + VW^a{}_{\widehat{b}}\, v^{\widehat{b}}\right). \tag{2.5}$$

We used (2.3) here. Dividing (2.5) by (2.4) and denoting

$$\mathcal{F}^a{}_b = VW^a{}_{\widehat{b}},$$

we find for the velocity with respect to the coordinate time

$$\frac{dx^a}{dt} = \mathcal{F}^a{}_b\, v^b + cK^a. \tag{2.6}$$

The Dirac equation in a curved spacetime reads

$$(i\hbar\gamma^\alpha D_\alpha - mc)\Psi = 0, \qquad \alpha = 0, 1, 2, 3. \tag{2.7}$$

This equation is invariant under the general transformations of the spacetime coordinates (under diffeomorphism), and is covariant under the local Lorentz transformations. Recall that the Dirac matrices γ^α are defined in local Lorentz (tetrad) frames and they have constant components. The spinor covariant derivatives are consistently defined in the gauge-theoretic approach [13–16] as

$$D_\alpha = e^i_\alpha D_i, \qquad D_i = \partial_i + \frac{iq}{\hbar}\, A_i + \frac{i}{4}\sigma^{\alpha\beta}\Gamma_{i\,\alpha\beta}. \tag{2.8}$$

Here the Lorentz connection is $\Gamma_i{}^{\alpha\beta} = -\Gamma_i{}^{\beta\alpha}$, and $\sigma^{\alpha\beta} = \frac{i}{2}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)$ are the generators of the local Lorentz transformations of the spinor field. For completeness, we assumed that the Dirac particle is charged and the electric charge q describes its coupling to the 4-potential of the electromagnetic field A_i. Note that the canonical dimension of the electromagnetic field strength 2-form $F = dA$ and of the electromagnetic 1-form $A = A_i dx^i$ is $[F] = [A] = [\hbar/q]$, see [45]. The gravitational and inertial effects (which are deeply related to each other in the framework of the gauge-theoretic approach) are encoded in coframe and connection in (2.7),(2.8); for the relevant discussion see Refs. [46–48] and references therein.

Using the parametrization of the general metric (2.1) with the tetrad (2.2) in the Schwinger gauge, we find explicitly the components of connection

$$\Gamma_{i\,\widehat{a}\widehat{0}} = \frac{c^2}{V}\, W^b{}_{\widehat{a}}\, \partial_b V\, e_i{}^{\widehat{0}} - \frac{c}{V}\, \mathcal{Q}_{(\widehat{a}\widehat{b})}\, e_i{}^{\widehat{b}}, \tag{2.9}$$

$$\Gamma_{i\,\widehat{a}\widehat{b}} = \frac{c}{V}\, \mathcal{Q}_{[\widehat{a}\widehat{b}]}\, e_i{}^{\widehat{0}} + \left(\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} + \mathcal{C}_{\widehat{a}\widehat{c}\widehat{b}} + \mathcal{C}_{\widehat{c}\widehat{b}\widehat{a}}\right) e_i{}^{\widehat{c}}. \tag{2.10}$$

Here we introduced the two useful objects:

$$\mathcal{Q}_{\widehat{a}\widehat{b}} = g_{\widehat{a}\widehat{c}} W^d{}_{\widehat{b}} \left(\frac{1}{c}\dot{W}^{\widehat{c}}{}_d + K^e \partial_e W^{\widehat{c}}{}_d + W^{\widehat{c}}{}_e \partial_d K^e\right), \tag{2.11}$$

$$\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = W^d{}_{\widehat{a}} W^e{}_{\widehat{b}} \partial_{[d} W^{\widehat{c}}{}_{e]}, \qquad \mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = g_{\widehat{c}\widehat{d}} \mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{d}}. \tag{2.12}$$

As usual, the dot $\dot{}$ denotes the partial derivative with respect to the coordinate time t. One can immediately recognize that $\mathcal{C}_{\widehat{a}\widehat{b}}{}^{\widehat{c}} = -\mathcal{C}_{\widehat{b}\widehat{a}}{}^{\widehat{c}}$ is the anholonomity object for the spatial triad $W^{\widehat{a}}{}_b$. The indices (that all run from 1 to 3) are raised and lowered with the help of the spatial part of the flat Minkowski metric, $g_{\widehat{a}\widehat{b}} = -\,\delta_{ab} = \mathrm{diag}(-1,-1,-1)$.

One can derive the Dirac equation from the action integral $I = \int d^4x\,\sqrt{-g}\,L$, with the Lagrangian (recall for the conjugate spinor $\overline{\Psi} := \Psi^\dagger\gamma^{\widehat{0}}$)

$$L = \frac{i\hbar}{2}\left(\overline{\Psi}\gamma^\alpha D_\alpha\Psi - D_\alpha\overline{\Psi}\gamma^\alpha\Psi\right) - mc\,\overline{\Psi}\Psi. \tag{2.13}$$

A direct check shows that, with (2.8)-(2.12) inserted, the Schrödinger form of the Dirac equation derived from this action involves a non-Hermitian Hamiltonian. However, this problem is fixed if we introduce a new wave function by

$$\psi = \left(\sqrt{-g}e_{\widehat{0}}^{0}\right)^{\frac{1}{2}}\,\Psi. \tag{2.14}$$

Such a non-unitary transformation also appears in the framework of the pseudo-Hermitian quantum mechanics [49, 50] (cf. [51]).

Variation of the action with respect to the *rescaled* wave function ψ yields the Dirac equation in Schrödinger form $i\hbar\frac{\partial\psi}{\partial t} = \mathcal{H}\psi$. The corresponding *Hermitian* Hamiltonian reads

$$\mathcal{H} = \beta mc^2 V + q\Phi + \frac{c}{2}\left(\pi_b\,\mathcal{F}^b{}_a\alpha^a + \alpha^a\mathcal{F}^b{}_a\pi_b\right) + \frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{K}\right) + \frac{\hbar c}{4}\left(\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma} - \Upsilon\gamma_5\right). \tag{2.15}$$

Here $\boldsymbol{K} = \{K^a\}$, and the kinetic momentum operator $\boldsymbol{\pi} = \{\pi_a\}$ with $\pi_a = -\,i\hbar\partial_a + qA_a = p_a + qA_a$ accounts for the interaction with the electromagnetic field $A_i = (\Phi, A_a)$. To remind the notation: $\beta = \gamma^{\hat{0}}, \boldsymbol{\alpha} = \{\alpha^a\}, \boldsymbol{\Sigma} = \{\Sigma^a\}$, where the 3-vector-valued Dirac matrices have their usual form; namely, $\alpha^a = \gamma^{\hat{0}}\gamma^a$ $(a,b,c,\dots = 1,2,3)$ and $\Sigma^1 = i\gamma^{\hat{2}}\gamma^{\hat{3}}, \Sigma^2 = i\gamma^{\hat{3}}\gamma^{\hat{1}}, \Sigma^3 = i\gamma^{\hat{1}}\gamma^{\hat{2}}$. We also introduced a pseudoscalar Υ and a 3-vector $\boldsymbol{\Xi} = \{\Xi_a\}$ by

$$\Upsilon = V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\Gamma_{\widehat{a}\widehat{b}\widehat{c}} = -V\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}}, \qquad \Xi_{\widehat{a}} = \frac{V}{c}\,\epsilon_{\widehat{a}\widehat{b}\widehat{c}}\,\Gamma_{\widehat{0}}{}^{\widehat{b}\widehat{c}} = \epsilon_{\widehat{a}\widehat{b}\widehat{c}}\,\mathcal{Q}^{\widehat{b}\widehat{c}}. \tag{2.16}$$

Note that we have fixed a number of small points with the signs and numeric factors, and one should be careful when comparing formulas above with the earlier results in [41]. For the static and stationary rotating configurations, the pseudoscalar invariant vanishes ($\epsilon^{\widehat{a}\widehat{b}\widehat{c}}\mathcal{C}_{\widehat{a}\widehat{b}\widehat{c}} = 0$), and thus the corresponding term was absent in the special cases considered earlier [40, 41]. But in general this term contributes to the Dirac Hamiltonian.

It is worthwhile to mention that the recent discussion [52] of the Dirac fermions in an arbitrary gravitational field is very different in that the non-Hermitian Hamiltonian is used in that work, in deep contrast to the explicitly Hermitian one (2.15).

III. THE FOLDY-WOUTHUYSEN TRANSFORMATION

In the previous section, we described the dynamics of the fermion in Dirac representation. The physical contents of the theory is however revealed in the Foldy-Wouthuysen representation. We will now construct the FW [53] Hamiltonian for the fermion moving in an arbitrary gravitational field described by the general metric (2.1). We start with the exact Dirac Hamiltonian (2.15) and apply the method developed in [54].

Just like before in our earlier work [40, 41], we do not make any assumptions and/or approximations for the functions $V, W^{\widehat{a}}{}_b, K^a$. The Planck constant $\hbar$ will be treated as the only small parameter. In accordance with this strategy, we retain in the FW Hamiltonian all the terms of the zero and first orders in $\hbar$. The leading nonvanishing terms of order $\hbar^2$ have been calculated in both nonrelativistic and weak field approximations in our previous works [37, 40, 41] for the more special cases. These terms describe the gravitational contact (Darwin) interaction.

A. General preliminaries

A generic Hamiltonian can be decomposed into operators that commute and anticommute with β:

$$\mathcal{H} = \beta\mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \beta\mathcal{M} = \mathcal{M}\beta, \quad \beta\mathcal{E} = \mathcal{E}\beta, \quad \beta\mathcal{O} = -\mathcal{O}\beta. \tag{3.1}$$

Here, the operators $\mathcal{M}, \mathcal{E}$ are even, and $\mathcal{O}$ is odd.

Foldy-Wouthuysen representation is obtained by means of the unitary transformation

$$\psi_{FW} = U\psi, \qquad \mathcal{H}_{FW} = U\mathcal{H}U^{-1} - i\hbar U\partial_t U^{-1}. \tag{3.2}$$

In arbitrary strong external fields, the following transformation operator can be used [54]:

$$U=\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}}\,\beta,\qquad U^{-1}=\beta\,\frac{\beta\epsilon+\beta\mathcal{M}-\mathcal{O}}{\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}}. \tag{3.3}$$

Here $\epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}$, and $U^{-1}=U^\dagger$ if $\mathcal{H}=\mathcal{H}^\dagger$. Applying (3.2), we obtain the explicit transformed Hamiltonian

$$\begin{aligned}\mathcal{H}'=\beta\epsilon+\mathcal{E}+\frac{1}{2T}\Bigg(&[T,[T,(\beta\epsilon+\mathcal{Z})]]+\beta\,[\mathcal{O},[\mathcal{O},\mathcal{M}]]\\ &-[\mathcal{O},[\mathcal{O},\mathcal{Z}]]-[(\epsilon+\mathcal{M}),[(\epsilon+\mathcal{M}),\mathcal{Z}]]-[(\epsilon+\mathcal{M}),[\mathcal{M},\mathcal{O}]]\\ &-\beta\,\{\mathcal{O},[(\epsilon+\mathcal{M}),\mathcal{Z}]\}+\beta\,\{(\epsilon+\mathcal{M}),[\mathcal{O},\mathcal{Z}]\}\Bigg)\frac{1}{T},\end{aligned} \tag{3.4}$$

where $\mathcal{Z}=\mathcal{E}-i\hbar\dfrac{\partial}{\partial t}$ and $T=\sqrt{(\beta\epsilon+\beta\mathcal{M}-\mathcal{O})^2}$. The square and curly brackets denote the commutator $[A,B]=AB-BA$ and the anticommutator $\{A,B\}=AB+BA$, respectively.

The Hamiltonian (3.4) still contains odd terms proportional to $\hbar$. We can write it as follows:

$$\mathcal{H}'=\beta\epsilon+\mathcal{E}'+\mathcal{O}',\quad \beta\mathcal{E}'=\mathcal{E}'\beta,\quad \beta\mathcal{O}'=-\mathcal{O}'\beta, \tag{3.5}$$

where $\epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}$. The even and odd parts are determined by

$$\mathcal{E}'=\frac{1}{2}\left(\mathcal{H}'+\beta\mathcal{H}'\beta\right)-\beta\epsilon,\qquad \mathcal{O}'=\frac{1}{2}\left(\mathcal{H}'-\beta\mathcal{H}'\beta\right). \tag{3.6}$$

Additional unitary transformation removes the odd part, so that the final approximate Hamiltonian reads [54]

$$\mathcal{H}_{FW}=\beta\epsilon+\mathcal{E}'+\frac{1}{4}\beta\left\{\mathcal{O}'^2,\frac{1}{\epsilon}\right\}. \tag{3.7}$$

For the case under consideration, we have explicitly

$$\mathcal{M}=mc^2V, \tag{3.8}$$

$$\mathcal{E}=q\Phi+\frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{K}\right)+\frac{\hbar c}{4}\,\boldsymbol{\Xi}\cdot\boldsymbol{\Sigma}, \tag{3.9}$$

$$\mathcal{O}=\frac{c}{2}\left(\pi_b\,\mathcal{F}^b{}_a\alpha^a+\alpha^a\mathcal{F}^b{}_a\pi_b\right)-\frac{\hbar c}{4}\,\Upsilon\gamma_5. \tag{3.10}$$

B. Foldy-Wouthuysen Hamiltonian and quantum dynamics

We now limit ourselves to the case when the electromagnetic field is switched off. The computations along the lines described in the previous subsection are straightforward, and after a lengthy algebra we obtain the Foldy-Wouthuysen Hamiltonian in the following form

$$\mathcal{H}_{FW} = \mathcal{H}^{(1)}_{FW} + \mathcal{H}^{(2)}_{FW}. \tag{3.11}$$

Here the two terms read, respectively,

$$\begin{aligned}\mathcal{H}^{(1)}_{FW} &= \beta\epsilon' + \frac{\hbar c^2}{16}\left\{\frac{1}{\epsilon'}, \left(2\epsilon^{cae}\Pi_e\{p_b, \mathcal{F}^d{}_c\partial_d\mathcal{F}^b{}_a\} + \Pi^a\{p_b, \mathcal{F}^b{}_a\Upsilon\}\right)\right\} \\ &\quad + \frac{\hbar mc^4}{4}\epsilon^{cae}\Pi_e\left\{\frac{1}{\mathcal{T}}, \{p_d, \mathcal{F}^d{}_c\mathcal{F}^b{}_a\partial_b V\}\right\}, \end{aligned}\tag{3.12}$$

$$\begin{aligned}\mathcal{H}^{(2)}_{FW} &= \frac{c}{2}\left(K^a p_a + p_a K^a\right) + \frac{\hbar c}{4}\,\Sigma_a\Xi^a \\ &\quad + \frac{\hbar c^2}{16}\left\{\frac{1}{\mathcal{T}}, \left\{\Sigma_a\{p_e, \mathcal{F}^e{}_b\}, \left\{p_f, \left[\epsilon^{abc}(\frac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c)\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. - \frac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\right]\right\}\right\}\right\}, \end{aligned}\tag{3.13}$$

$$\epsilon' = \sqrt{m^2c^4V^2 + \frac{c^2}{4}\delta^{ac}\{p_b, \mathcal{F}^b{}_a\}\{p_d, \mathcal{F}^d{}_c\}}, \qquad \mathcal{T} = 2\epsilon'^2 + \{\epsilon', mc^2V\}. \tag{3.14}$$

Let us derive the equation of motion of spin. As usual, we introduce the polarization operator $\mathbf{\Pi} = \beta\mathbf{\Sigma}$, and the corresponding dynamical equation is obtained from its commutator with the FW Hamiltonian. The computation is straightforward and we find

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \mathbf{\Pi}] = \mathbf{\Omega}_{(1)} \times \mathbf{\Sigma} + \mathbf{\Omega}_{(2)} \times \mathbf{\Pi}. \tag{3.15}$$

Here the 3-vectors $\mathbf{\Omega}_{(1)}$ and $\mathbf{\Omega}_{(2)}$ are the operators of the angular velocity of spin precessing in the exterior classical gravitational field. Their components read explicitly as follows:

$$\begin{aligned}\Omega^a_{(1)} &= \frac{mc^4}{2}\left\{\frac{1}{\mathcal{T}}, \{p_e, \epsilon^{abc}\mathcal{F}^e{}_b\mathcal{F}^d{}_c\partial_d\, V\}\right\} \\ &\quad + \frac{c^2}{8}\left\{\frac{1}{\epsilon'}, \{p_e, (2\epsilon^{abc}\mathcal{F}^d{}_b\partial_d\mathcal{F}^e{}_c + \delta^{ab}\mathcal{F}^e{}_b\,\Upsilon)\}\right\}, \end{aligned}\tag{3.16}$$

and

$$\begin{aligned}\Omega^a_{(2)} &= \frac{\hbar c^2}{8}\left\{\frac{1}{\mathcal{T}}, \left\{\{p_e, \mathcal{F}^e{}_b\}, \left\{p_f, \left[\epsilon^{abc}(\frac{1}{c}\dot{\mathcal{F}}^f{}_c - \mathcal{F}^d{}_c\partial_d K^f + K^d\partial_d\mathcal{F}^f{}_c)\right.\right.\right.\right. \\ &\quad \left.\left.\left.\left. - \frac{1}{2}\mathcal{F}^f{}_d\left(\delta^{db}\Xi^a - \delta^{da}\Xi^b\right)\right]\right\}\right\}\right\} + \frac{c}{2}\,\Xi^a. \end{aligned}\tag{3.17}$$

One may notice that the two different matrices, $\mathbf{\Sigma}$ and $\mathbf{\Pi}$, appear on the right-hand side of Eq. (3.15). This is explained by the fact that the vector $\mathbf{\Omega}_{(1)}$ contains odd number of components of the momentum operator, whereas the vector $\boldsymbol{\Omega}_{(2)}$ contains even number of p_a. Actually, both $\boldsymbol{\Omega}_{(1)}$ and $\boldsymbol{\Omega}_{(2)}$ depend only on the combination $\mathcal{F}^b{}_a p_b$. However, the velocity operator is proportional to an additional β factor and is equal to $v_a = \beta c^2 \mathcal{F}^b{}_a p_b/\epsilon'$, as we demonstrate below, see (3.24). As a result, the operator $\boldsymbol{\Omega}_{(1)}$ also acquires an additional β factor [40], when it is rewritten in terms of the velocity operator $\boldsymbol{v}$. Note also that in the FW representation only upper part of β proportional the unit matrix is relevant. Therefore, the appearance of β does not lead to any physical effects at least until antiparticles are considered (which would require special investigations).

We now use the general results above to obtain the corresponding semiclassical expressions by evaluating all anticommutators and neglecting the powers of $\hbar$ higher than 1. Then equations (3.15)-(3.17) yield the following explicit semiclassical equations describing the motion of the average spin (as usual, vector product is defined by $\{\boldsymbol{A}\times\boldsymbol{B}\}_a = \epsilon_{abc}A^bB^c$):

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{s} = (\boldsymbol{\Omega}_{(1)} + \boldsymbol{\Omega}_{(2)})\times\boldsymbol{s}, \tag{3.18}$$

$$\Omega^a_{(1)} = \frac{c^2}{\epsilon'}\mathcal{F}^d{}_c p_d\left(\frac{1}{2}\Upsilon\delta^{ac} - \epsilon^{aef}V\mathcal{C}_{ef}{}^c + \frac{\epsilon'}{\epsilon' + mc^2V}\epsilon^{abc}W^e{}_{\hat{b}}\partial_e V\right), \tag{3.19}$$

$$\Omega^a_{(2)} = \frac{c}{2}\,\Xi^a - \frac{c^3}{\epsilon'(\epsilon' + mc^2V)}\,\epsilon^{abc}Q_{(bd)}\delta^{dn}\mathcal{F}^k{}_n p_k \mathcal{F}^l{}_c p_l, \tag{3.20}$$

where, in the semiclassical limit,

$$\epsilon' = \sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,p_a\,p_b}\,. \tag{3.21}$$

We can substitute the results obtained into the FW Hamiltonian (3.11) and recast the latter in a compact and transparent form in terms of the precession angular velocities $\boldsymbol{\Omega}_{(1)}, \boldsymbol{\Omega}_{(2)}$:

$$\mathcal{H}_{FW} = \beta\epsilon' + \frac{c}{2}\left(\boldsymbol{K}\cdot\boldsymbol{p} + \boldsymbol{p}\cdot\boldsymbol{K}\right) + \frac{\hbar}{2}\boldsymbol{\Pi}\cdot\boldsymbol{\Omega}_{(1)} + \frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\Omega}_{(2)}. \tag{3.22}$$

We can use (3.22) to derive the velocity operator in the semiclassical approximation:

$$\frac{dx^a}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, x^a] = \beta\,\frac{\partial\epsilon'}{\partial p_a} + cK^a = \beta\,\frac{c^2}{\epsilon'}\,\mathcal{F}^a{}_b\delta^{bc}\mathcal{F}^d{}_c p_d + cK^a. \tag{3.23}$$

Comparing this with the relation between the holonomic and anholonomic components of the velocity, (2.6), we find the velocity operator in the Schwinger frame (2.2):

$$\beta\,\frac{c^2}{\epsilon'}\,\mathcal{F}^b{}_a p_b = v_a. \tag{3.24}$$

This immediately yields $\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d p_a p_b = (\epsilon')^2 v^2/c^2$. Using this in (3.21), we have $(\epsilon')^2 = m^2c^4V^2 + (\epsilon')^2v^2/c^2$, and thus we find

$$\epsilon' = \gamma\, mc^2\, V. \tag{3.25}$$

Equations (3.24) and (3.25) are crucial for establishing the full agreement of the quantum and classical dynamics of spin. In particular, using (3.24) and (3.25), we find

$$\frac{\epsilon'}{\epsilon' + mc^2V} = \frac{\gamma}{1+\gamma}, \qquad \frac{c^3}{\epsilon'(\epsilon' + mc^2V)}\,\mathcal{F}^b{}_a p_b \mathcal{F}^d{}_c p_d = \frac{\gamma}{1+\gamma}\,\frac{v_a v_c}{c}. \tag{3.26}$$

C. Quantum-mechanical equations of particle dynamics

We now turn to the analysis of the motion of the quantum particle in the gravitational field. The dynamics of spin is described in an anholonomic frame. For consistency, we will use an anholonomic frame description for the particle dynamics, too. The Schwinger gauge with $e^0_{\widehat{a}} = 0$ simplifies the equation for the force operator which is given by

$$\begin{aligned} F_{\widehat{a}} = \frac{dp_{\widehat{a}}}{dt} &= \frac{1}{2}\frac{d}{dt}\left\{e^b_{\widehat{a}}, p_b\right\} = \frac{1}{2}\left\{\frac{dW^b{}_{\widehat{a}}}{dt}, p_b\right\} + \frac{1}{2}\left\{W^b{}_{\widehat{a}}, \frac{dp_b}{dt}\right\} \\ &= \frac{1}{2}\left\{\dot{W}^b{}_{\widehat{a}}, p_b\right\} + \frac{i}{2\hbar}\left\{[\mathcal{H}_{FW}, W^b{}_{\widehat{a}}], p_b\right\} - \frac{1}{2}\left\{W^b{}_{\widehat{a}}, \partial_b\mathcal{H}_{FW}\right\}. \end{aligned} \tag{3.27}$$

Here as before the partial derivative with respect to the coordinate time is denoted by the dot, in particular, $\dot{W}^b{}_{\widehat{a}} := \partial_t W^b{}_{\widehat{a}}$.

The explicit expression for the force operator reads

$$F_{\widehat{a}} = \frac{1}{2}\left\{\dot{W}^b{}_{\widehat{a}}, p_b\right\} + \frac{1}{4}\left\{p_b, \left\{\frac{\partial\mathcal{H}_{FW}}{\partial p_c}, \partial_c W^b{}_{\widehat{a}}\right\}\right\} - \frac{1}{2}\left\{W^b{}_{\widehat{a}}, \partial_b\mathcal{H}_{FW}\right\}, \tag{3.28}$$

$$\frac{\partial\mathcal{H}_{FW}}{\partial p_c} = \beta\frac{c^2}{4}\delta^{ad}\left\{\frac{1}{\epsilon'}, \left\{p_b, \mathcal{F}^b{}_a\mathcal{F}^c{}_d\right\}\right\} + cK^c + \frac{\hbar}{2}\mathfrak{T}^c, \tag{3.29}$$

where we introduced the following compact notation

$$\mathfrak{T}^c = \frac{\partial\mathcal{U}}{\partial p_c}, \qquad \mathcal{U} := \mathbf{\Pi}\cdot\mathbf{\Omega}_{(1)} + \mathbf{\Sigma}\cdot\mathbf{\Omega}_{(2)}. \tag{3.30}$$

Corrections due to the noncommutativity of operators are of order of $\hbar^2$ and can be neglected in (3.28). Let us split the total force operator into the terms of the zeroth and first orders in the Planck constant:

$$F_{\widehat{a}} = F^{(0)}_{\widehat{a}} + F^{(1)}_{\widehat{a}}. \tag{3.31}$$

The zeroth order terms read as follows

$$F^{(0)}_{\widehat{a}} = \frac{1}{2}\left\{\dot{W}^b{}_{\widehat{a}}, p_b\right\} - \frac{1}{2}\left\{W^b{}_{\widehat{a}}, \partial_b\left[\beta\epsilon' + \frac{c}{2}\left(K^a p_a + p_a K^a\right)\right]\right\}$$
$$+\frac{1}{4}\left\{p_b, \left\{\left(\beta\frac{c^2}{4}\delta^{ad}\left\{\frac{1}{\epsilon'}, \{p_b, \mathcal{F}^b{}_a\mathcal{F}^c{}_d\}\right\} + cK^c\right), \partial_c W^b{}_{\widehat{a}}\right\}\right\}. \tag{3.32}$$

These terms describe the influence of the gravitational field on the particle without taking into account its internal structure. The first term in Eq. (3.32) is important for the motion of the particle in nonstationary spacetimes, for example, in cosmological context. The next term describes the Newtonian force, the related relativistic corrections, and the Coriolis-like force proportional to $\boldsymbol{K}$. The last term also contributes to the relativistic corrections to the force acting in static spacetimes that arise in addition to the velocity-independent Newtonian force.

All the terms of the first order in the Planck constant are proportional to the spin operators and therefore they collectively represent the quantum-mechanical counterpart of the Mathisson force (which is an analogue of the Stern-Gerlach force in electrodynamics). This force is given by, recall the notation (3.30),

$$F^{(1)}_{\widehat{a}} = \frac{\hbar}{8}\left\{p_b, \left\{\mathfrak{T}^c, \partial_c W^b{}_{\widehat{a}}\right\}\right\} - \frac{\hbar}{4}\left\{W^b{}_{\widehat{a}}, \partial_b\mathcal{U}\right\}. \tag{3.33}$$

In the next section, we will demonstrate the agreement between the quantum-mechanical and the classical equations of particle dynamics.

Eqs. (3.32) and (3.33) perfectly reproduce all previously obtained quantum-mechanical results [37, 38, 40, 41]. In order to illustrate this, let us find the force on the spinning particle in the metric [46] of an arbitrarily moving noninertial (accelerated and rotating) observer:

$$V = 1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}, \qquad W^{\widehat{a}}{}_b = \delta^a_b, \qquad K^a = -\frac{1}{c}\left(\boldsymbol{\omega}\times\boldsymbol{r}\right)^a. \tag{3.34}$$

The FW Hamiltonian for this metric was derived in [41]. It reads:

$$\mathcal{H}_{FW} = \frac{\beta}{2}\left\{\left(1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\right), \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}\right\} - \boldsymbol{\omega}\cdot(\boldsymbol{r}\times\boldsymbol{p})$$
$$+\frac{\hbar}{2}\boldsymbol{\Pi}\cdot\frac{\boldsymbol{a}\times\boldsymbol{p}}{mc^2(\gamma+1)} - \frac{\hbar}{2}\boldsymbol{\Sigma}\cdot\boldsymbol{\omega}, \tag{3.35}$$

where the object that has the meaning of the Lorentz factor is defined by

$$\gamma = \frac{\sqrt{m^2c^4 + c^2\boldsymbol{p}^2}}{mc^2}. \tag{3.36}$$

Using the FW Hamiltonian (3.35) in (3.28) and (3.29) yields the explicit force

$$
\begin{aligned}
F_{\widehat{a}} &= -\frac{\beta}{c^2}\, a_a \sqrt{m^2c^4 + c^2\boldsymbol{p}^2} - (\boldsymbol{\omega}\times\boldsymbol{p})_a \\
&= \beta m\gamma\left(-\boldsymbol{a} + \boldsymbol{v}\times\boldsymbol{\omega}\right)_a . \qquad (3.37)
\end{aligned}
$$

Here we used (3.36) and the relation between the operators of momentum and velocity $p_{\widehat{a}} = e^b_{\widehat{a}} p_b = \beta\gamma m v_a$ which is recovered from (3.24). One can straightforwardly verify that the usual structure of the inertial forces (in particular, the Coriolis and centrifugal pieces) is encoded in the force (3.37), see the corresponding computation of the coordinate acceleration operator in [41].

For the metric (3.34), the spacetime curvature vanishes. As a result, the curvature- and spin-dependent Mathisson force is zero. In the general case, the Mathisson force is nontrivial, and the validity of the equivalence principle is an open question (see, e.g., Ref. [55]). In a separate publication, we will analyse the possible generalization of the equivalence principle for spinning particles, making use of the force framework developed here. As a preliminary step, we refer to [41] where we evaluated the quantum force for the weak gravitational field and recovered the linearized Mathisson force, thus confirming the earlier results [56, 57].

Any theory based on the Dirac equation can reproduce only a certain reduced form of the equation of spin motion. The formal reason is the absence in the Lagrangian and the Hamiltonian of the terms bilinear in the spin matrices because their product can always be simplified: $\Sigma^a\Sigma^b = \delta^{ab} + i\epsilon^{abc}\Sigma^c$. As a result, the equation of spin motion of a Dirac particle cannot contain such terms. In quantum mechanics of particles with higher spins ($s > 1/2$) as well as in the classical gravity, the terms bilinear in spin cannot be reduced and the general MP equations [58, 59] should be used.

IV. CLASSICAL SPINNING PARTICLES

A. Mathisson-Papapetrou approach

The motion of classical spinning particles in the gravitational field can be consistently described by the generally covariant MP theory [58, 59], for the recent discussion see also [60–63]). In this framework, a test particle is characterized by the 4-velocity U^α and the tensor of spin $S^{\alpha\beta} = -S^{\beta\alpha}$. The total 4-momentum is not collinear with the velocity, in general. In [64], a different noncovariant approach was developed, in which the main dynamical variable is the 3-dimensional spin defined in the rest frame of a particle. In our previous work [37–41] we have used the MP theory, and demonstrated its consistency with the noncovariant approach.

The analysis of the general MP equations is a difficult task [62] and the exact solutions are not available even for the simple spacetime geometries. The knowledge of the symmetries of the gravitational field, i.e., of the corresponding Killing vectors, significantly helps in the integration of the equations of motion, as can be demonstrated [65] for the de Sitter spacetime, in particular. However, in the absence of the symmetries, various approximation schemes were developed to find solutions of MP equations of motion. Following [60], we neglect the second order spin effects, so that the MP system reduces to

$$\frac{DU^\alpha}{d\tau} = f^\alpha_{\rm m} = -\,\frac{1}{2m}\,S^{\mu\nu}U^\beta R_{\mu\nu\beta}{}^\alpha, \tag{4.1}$$

$$\frac{DS^{\alpha\beta}}{d\tau} = \frac{U^\alpha U_\gamma}{c^2}\frac{DS^{\gamma\beta}}{d\tau} - \frac{U^\beta U_\gamma}{c^2}\frac{DS^{\gamma\alpha}}{d\tau}. \tag{4.2}$$

On the right-hand side of (4.1) we have the Mathisson force $f^\alpha_{\rm m}$ that depends on the Riemann curvature tensor $R_{\mu\nu\beta}{}^\alpha$ of spacetime. The tensor of spin satisfies the Frenkel condition $U_\alpha S^{\alpha\beta} = 0$ and gives rise to the vector of spin

$$S_\alpha = \frac{1}{2}\epsilon_{\alpha\beta\gamma}S^{\beta\gamma}. \tag{4.3}$$

Here we use the totally antisymmetric tensor

$$\epsilon_{\alpha\beta\gamma} = \frac{1}{c}\eta_{\alpha\beta\gamma\delta}U^\delta, \tag{4.4}$$

constructed from the Levi-Civita tensor $\eta_{\alpha\beta\gamma\delta}$. The relation (4.3) can be inverted

$$S^{\alpha\beta} = -\,\epsilon^{\alpha\beta\gamma}S_\gamma \tag{4.5}$$

with the help of the identity

$$\epsilon^{\alpha\beta\gamma}\epsilon_{\mu\nu\gamma} = P^\alpha_\nu P^\beta_\mu - P^\alpha_\mu P^\beta_\nu, \tag{4.6}$$

where $P^\alpha_\mu = \delta^\alpha_\mu - \frac{1}{c^2}U^\alpha U_\mu$ is the projector on the rest frame of the particle.

Using the definition (4.3), we rewrite the equation (4.2) in an alternative form

$$\frac{DS_\alpha}{d\tau} = \frac{U_\alpha U^\beta}{c^2}\frac{DS_\beta}{d\tau} = -\,\frac{1}{c^2}U_\alpha f^\beta_{\rm m}S_\beta. \tag{4.7}$$

With the help of the projectors and antisymmetric tensor, one can decompose the curvature tensor into the three irreducible pieces

$$\mathbb{E}_{\alpha\beta} = \frac{U^\mu U^\nu}{c^2} R_{\alpha\mu\beta\nu}, \tag{4.8}$$

$$\mathbb{M}^{\alpha\beta} = \frac{1}{4}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma} R_{\mu\nu\rho\sigma}, \tag{4.9}$$

$$\mathbb{B}_\alpha{}^\beta = \frac{U_\gamma}{2c}\epsilon_{\alpha\mu\nu} R^{\beta\gamma\mu\nu}. \tag{4.10}$$

By construction, these tensors satisfy the orthogonality conditions $\mathbb{E}_{\alpha\beta}U^\beta = 0$, $\mathbb{M}^{\alpha\beta}U_\beta = 0$, $\mathbb{B}_\alpha{}^\beta U_\beta = 0$, $\mathbb{B}_\alpha{}^\beta U^\alpha = 0$. Taking into account the obvious symmetry $\mathbb{E}_{\alpha\beta} = \mathbb{E}_{\beta\alpha}$ and $\mathbb{M}^{\alpha\beta} = \mathbb{M}^{\beta\alpha}$, we have $6 + 6 + 8 = 20$ independent components for these objects. The curvature decomposition reads explicitly

$$\begin{aligned} R^{\alpha\beta\mu\nu} = & \frac{1}{c^2}\left(U^\alpha U^\mu \mathbb{E}^{\beta\nu} - U^\beta U^\mu \mathbb{E}^{\alpha\nu} - U^\alpha U^\nu \mathbb{E}^{\beta\mu} + U^\beta U^\nu \mathbb{E}^{\alpha\mu}\right) + \epsilon^{\alpha\beta\gamma}\epsilon^{\mu\nu\lambda}\mathbb{M}_{\gamma\lambda} \\ & + \frac{1}{c}\left[\epsilon^{\alpha\beta\gamma}\left(U^\mu \mathbb{B}_\gamma{}^\nu - U^\nu \mathbb{B}_\gamma{}^\mu\right) + \epsilon^{\mu\nu\gamma}\left(U^\alpha \mathbb{B}_\gamma{}^\beta - U^\beta \mathbb{B}_\gamma{}^\alpha\right)\right]. \end{aligned} \tag{4.11}$$

As a result, we rewrite the Mathisson force as

$$f^\alpha_{\rm m} = \frac{c}{2m}\mathbb{B}_\beta{}^\alpha S^\beta. \tag{4.12}$$

The physical spin is defined in the rest frame of a particle where the 4-velocity reduces to $u^\alpha = (1, \mathbf{0}) = \delta^\alpha_0$. The local reference frame and the rest frame are related by the Lorentz transformation such that $U^\alpha = \Lambda^\alpha{}_\beta u^\beta$. Recalling $U^\alpha = (\gamma, \gamma v^a)$, the Lorentz matrix reads explicitly

$$\Lambda^\alpha{}_\beta = \left(\begin{array}{c|c} \gamma & \gamma v_b/c^2 \\ \hline \gamma v^a & \delta^a_b + (\gamma - 1)v^a v_b/v^2 \end{array}\right), \tag{4.13}$$

with the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$, where $v^2 = \delta_{ab}v^a v^b$.

The physical spin is then $s^\alpha = (\Lambda^{-1})^\alpha{}_\beta S^\beta$, hence $s^\alpha = (0, \boldsymbol{s})$. We rewrite equation (4.7) as $\dfrac{dS^\alpha}{d\tau} = \Phi^\alpha{}_\beta S^\beta$, with $\Phi^\alpha{}_\beta = -U^i\Gamma_{i\beta}{}^\alpha + \dfrac{1}{c^2}(f^\alpha_{\rm m}U_\beta - f_{\beta{\rm m}}U^\alpha)$. From this we find the equation of motion of the physical spin:

$$\frac{ds^\alpha}{d\tau} = \Omega^\alpha{}_\beta s^\beta, \tag{4.14}$$

$$\Omega^\alpha{}_\beta = (\Lambda^{-1})^\alpha{}_\gamma \Phi^\gamma{}_\delta \Lambda^\delta{}_\beta - (\Lambda^{-1})^\alpha{}_\gamma \frac{d}{d\tau}\Lambda^\gamma{}_\beta. \tag{4.15}$$

Noticing that with respect to the coordinate basis the 4-velocity is $U^i = \gamma e^i_{\widehat{0}} + \gamma v^a\, e^i_{\widehat{a}}$, we recast the MP system (4.1) and (4.14) into the 3-vector form

$$\frac{d\gamma}{d\tau} = \frac{\gamma}{c^2}\,\boldsymbol{v}\cdot\widehat{\boldsymbol{\mathcal{E}}}, \tag{4.16}$$

$$\frac{d(\gamma\boldsymbol{v})}{d\tau} = \gamma\left(\widehat{\boldsymbol{\mathcal{E}}} + \boldsymbol{v}\times\boldsymbol{\mathcal{B}}\right), \tag{4.17}$$

$$\frac{d\boldsymbol{s}}{d\tau} = \boldsymbol{\Omega}\times\boldsymbol{s}. \tag{4.18}$$

Here using (2.1) and (4.12), we introduced the objects that can be called the generalized gravitoelectric and gravitomagnetic fields:

$$\mathcal{E}^a = \frac{\gamma}{V}\delta^{ac}\left(c\mathcal{Q}_{(\widehat{c}\widehat{b})}v^b - c^2\,W^b{}_{\widehat{c}}\,\partial_b V\right), \tag{4.19}$$

$$\mathcal{B}^a = \frac{\gamma}{V}\left(-\frac{c}{2}\,\Xi^a - \frac{1}{2}\Upsilon\,v^a + \epsilon^{abc}V\mathcal{C}_{bc}{}^d v_d\right), \tag{4.20}$$

$$\widehat{\boldsymbol{\mathcal{E}}}^a = \boldsymbol{\mathcal{E}}^a + \frac{c}{2m\gamma}\mathbb{B}_b{}^a\left(s^b - \frac{\gamma}{\gamma+1}\frac{v^b v_c}{c^2}s^c\right). \tag{4.21}$$

The components of the angular velocity of the spin precession $\boldsymbol{\Omega} = \left\{-\frac{1}{2}\epsilon^{abc}\Omega_{bc}\right\}$ are obtained from (4.15):

$$\boldsymbol{\Omega} = -\boldsymbol{\mathcal{B}} + \frac{\gamma}{\gamma+1}\,\frac{\boldsymbol{v}\times\boldsymbol{\mathcal{E}}}{c^2}. \tag{4.22}$$

Alternatively, we can explicitly write the precession velocity components with the help of (2.9) and (2.10) as [40, 64]

$$\Omega_{\widehat{a}} = \epsilon_{abc}\,U^i\left(\frac{1}{2}\Gamma_i{}^{\widehat{c}\widehat{b}} + \frac{\gamma}{\gamma+1}\,\Gamma_{i\widehat{0}}{}^{\widehat{b}}v^{\widehat{c}}/c^2\right). \tag{4.23}$$

Finally, substituting (4.19) and (4.20) into (4.22), we obtain the *exact classical formula* for the angular velocity of the spin precession in an arbitrary gravitational field:

$$\begin{aligned}\Omega^{\widehat{a}} = \frac{\gamma}{V}\Bigg(&\frac{1}{2}\Upsilon\,v^{\widehat{a}} - \epsilon^{abc}V\mathcal{C}_{\widehat{b}\widehat{c}}{}^{d}v_{\widehat{d}} + \frac{\gamma}{\gamma+1}\epsilon^{abc}W^d{}_{\widehat{b}}\,\partial_d V\,v_{\widehat{c}}\\ &+\frac{c}{2}\,\Xi^{\widehat{a}} - \frac{\gamma}{\gamma+1}\epsilon^{abc}\mathcal{Q}_{(\widehat{b}\widehat{d})}\frac{v^{\widehat{d}}v_{\widehat{c}}}{c}\Bigg).\end{aligned} \tag{4.24}$$

The terms in the first line are linear in the 4-velocity of the particle, whereas the terms in the second line contain the even number of the velocity factors.

As compared to the precession of the quantum spin described by $\boldsymbol{\Omega}^{(1)}$ and $\boldsymbol{\Omega}^{(2)}$ using the coordinate time, the classical spin precession velocity $\Omega^{\widehat{a}}$ contains an extra factor

$$\frac{dt}{d\tau} = U^0 = \frac{\gamma}{V}, \tag{4.25}$$

since the classical dynamics is parameterized using the proper time.

It is worthwhile to notice that the equations of motion of a particle (4.16) and (4.17) have a remarkably simple form of the motion of a relativistic charged particle under the action of the Lorentz force. It is interesting to mention a certain asymmetry: the Mathisson force (4.12), that depends on the spin and the curvature of spacetime, contributes only to the gravitoelectric field (4.21) but not to the gravitomagnetic one. Using (4.16) in (4.17), we can recast the latter into the dynamical equation

$$\frac{d\boldsymbol{v}}{d\tau} = \widehat{\boldsymbol{\mathcal{E}}} - \frac{\boldsymbol{v}(\boldsymbol{v}\cdot\widehat{\boldsymbol{\mathcal{E}}})}{c^2} + \boldsymbol{v}\times\boldsymbol{\mathcal{B}}. \tag{4.26}$$

Let us consider the motion of the classical particle in the metric of a noninertial observer (3.34). Computing the gravitoelectric and gravitomagnetic fields is straightforward: $\widehat{\boldsymbol{\mathcal{E}}} = \boldsymbol{\mathcal{E}} = -\frac{\gamma}{V}\,\boldsymbol{a}$, and $\boldsymbol{\mathcal{B}} = \frac{\gamma}{V}\,\boldsymbol{\omega}$. As a result,

$$\frac{d(\gamma\boldsymbol{v})}{dt} = \gamma\left(-\,\boldsymbol{a} + \boldsymbol{v}\times\boldsymbol{\omega}\right), \tag{4.27}$$

where we changed from the proper time parametrization to the coordinate time using (4.25). As we see, the classical (4.27) and the quantum (3.37) forces are the same.

Finally, making use of (3.24) and (3.25), we conclude that the classical equation of the spin motion (4.22) agrees with the quantum equation (3.15) and with the semiclassical one (3.18). Thus, the classical and the quantum theories of the spin motion in gravity are in complete agreement. This is now verified for the *arbitrary gravitational field* configurations. We thus confirm and extend our previous results obtained for the weak fields [40] and for special strong field configurations [41].

B. Hamiltonian approach

It is instructive to compare the classical and quantum Hamiltonians of a spinning particle. In order to do this, one can start from the classical Hamiltonian of a spinless relativistic point particle (with an electric charge q, in general). The action has the well-known form

$$I = -\int mc^2 d\tau + qA_i dx^i = -\int \left[mc\left(g_{ij}U^iU^j\right)^{1/2} + qA_iU^i\right] d\tau. \tag{4.28}$$

In order to avoid working with the constrained system, we will use the deparatmetrized formulation. With the 3-velocity $v^a = dx^a/dt = U^a/U^0$, we then recast the action into

$I = \int \mathcal{L} dt$, where

$$\mathcal{L} = -mc\left(g_{00} + 2g_{0a}v^a + g_{ab}v^a v^b\right)^{1/2} - qA_0 - qA_b v^b. \tag{4.29}$$

The canonical momentum is

$$p_a = \frac{\partial \mathcal{L}}{\partial v^a} = -\frac{mc(g_{0a} + g_{ab}v^b)}{(g_{00} + 2g_{0a}v^a + g_{ab}v^a v^b)^{1/2}} - qA_a. \tag{4.30}$$

Inverting, we find velocity in terms of momentum $\pi_a = p_a + qA_a$

$$v^a = \frac{g^{0a}}{g^{00}} - \frac{\tilde{g}^{ab}\pi_b}{[g^{00}(m^2c^2 - \tilde{g}^{ab}\pi_a\pi_b)]^{1/2}}, \qquad \tilde{g}^{ab} = g^{ab} - \frac{g^{0a}g^{0b}}{g^{00}}. \tag{4.31}$$

As a result, the classical Hamiltonian reads (we fix some sign errors of [67] here):

$$\mathcal{H}_{class} = p_a v^a - \mathcal{L} = \left(\frac{m^2c^2 - \tilde{g}^{ab}\pi_a\pi_b}{g^{00}}\right)^{1/2} + \frac{g^{0a}\pi_a}{g^{00}} + qA_0. \tag{4.32}$$

For the contravariant components of the general metric (2.1) we have $g^{ij} = e^i_\alpha e^j_\beta g^{\alpha\beta} = \frac{1}{c^2} e^i_{\hat{0}} e^j_{\hat{0}} - e^i_{\hat{c}} e^j_{\hat{d}} \delta^{cd}$. Thus explicitly, using (2.3):

$$g^{00} = \frac{1}{c^2V^2}, \qquad g^{0a} = \frac{K^a}{cV^2}, \qquad g^{ab} = \frac{1}{V^2}\left(K^aK^b - \mathcal{F}^a{}_c\mathcal{F}^b{}_d\delta^{cd}\right). \tag{4.33}$$

As a result, the classical Hamiltonian (4.32) reads:

$$\mathcal{H}_{class} = \sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,\pi_a\,\pi_b} + c\boldsymbol{K}\cdot\boldsymbol{\pi} + q\Phi. \tag{4.34}$$

Now, let us discuss a generalization of the Hamiltonian theory with spin included. In order to take into account the spin correctly, in a Cosserat type approach a material frame (of four linearly independent vectors) is attached to a particle, thus modelling its internal rotational degrees of freedom. We denote it h^i_α.

Such a material frame does not coincide with the spacetime frame, $h^i_\alpha \neq e^i_\alpha$. In particular, the zeroth leg is given by particle's 4-velocity

$$h^i_{\hat{0}} = U^i. \tag{4.35}$$

Any two orthonormal frames are related by a Lorentz transformation, $h^i_\alpha = e^i_\beta \Lambda^\beta{}_\alpha$. The condition (4.35) means that the Lorentz matrix $\Lambda^\beta{}_\alpha$ brings one to a local reference frame $U^\alpha = \Lambda^\alpha{}_\beta u^\beta$ in which the particle is at rest, i.e., $u^\alpha = \delta^\alpha_{\hat{0}}$. This is straightforwardly demonstrated: $U^i = e^i_\alpha U^\alpha = e^i_\alpha \Lambda^\alpha{}_\beta u^\beta = h^i_\alpha u^\alpha = h^i_{\hat{0}}$. The corresponding Lorentz transformation is explicitly given by (4.13).

The standard way to take the dynamics of spin into account [68–70] is to amend the classical Hamiltonian by the term $\frac{1}{2}S^{ij}\Omega_{ij}$ with

$$\Omega^i{}_j := h^i_\alpha \frac{D}{d\tau} h^\alpha_j = h^i_\alpha U^k \nabla_k h^\alpha_j = h^i_\alpha U^k \left(\partial_k h^\alpha_j - \Gamma_{kj}{}^l h^\alpha_l\right). \tag{4.36}$$

Rewriting everything in terms of the objects in particle's rest frame, $S^{\alpha\beta} = h^\alpha_i h^\beta_j S^{ij}$ and $\Omega^\alpha{}_\beta = h^\alpha_i h^j_\beta \Omega^i{}_j$, we find

$$\frac{1}{2}S^{ij}\Omega_{ij} = \frac{1}{2}S^{\alpha\beta}\Omega_{\alpha\beta} = \boldsymbol{s}\cdot\boldsymbol{\Omega}. \tag{4.37}$$

Here we recover the precession velocity vector (4.23).

The resulting complete Hamiltonian has the structure that was proposed in the framework of the general discussion in the Ref. [64]:

$$\mathcal{H}_{class} = \sqrt{m^2c^4V^2 + c^2\delta^{cd}\mathcal{F}^a{}_c\,\mathcal{F}^b{}_d\,\pi_a\,\pi_b} + c\boldsymbol{K}\cdot\boldsymbol{\pi} + q\Phi + \boldsymbol{s}\cdot\boldsymbol{\Omega}. \tag{4.38}$$

In the general case, $\boldsymbol{\Omega}$ should include both electromagnetic and gravitational contributions.

The obvious similarity of quantum (3.22) and classical (4.38) Hamiltonians is another demonstration of complete agreement of the quantum-mechanical and classical equations of motion discussed in the previous subsection. The consistency between the classical Hamiltonian dynamics and the quantum-mechanical equations of particle dynamics derived in Sec. III C is also confirmed by the computation of the force. Switching off the electromagnetic field, we find the classical equation for the force

$$F^{class}_{\hat{a}} = p_b\dot{W}^b{}_{\hat{a}} + p_b\,\frac{\partial\mathcal{H}_{class}}{\partial p_c}\,\partial_c W^b{}_{\hat{a}} - W^b{}_{\hat{a}}\partial_b\mathcal{H}_{class}. \tag{4.39}$$

As we see, the equations (3.28) and (4.39) completely agree. In particular, rewriting the spin-dependent part in Eq.(3.28) in terms of the spin operator, $\boldsymbol{s} = \hbar\boldsymbol{\Sigma}/2$, shows the consistency of the corresponding parts in the two equations.

V. CONCLUSIONS

This paper continues the study of the motion of the quantum and classical Dirac fermion particles with spin 1/2 on a curved spacetime. Generalizing our earlier findings in [37–41] obtained for the weak fields and for the special static and stationary field configurations, we now consider the case of an absolutely arbitrary spacetime metric. The convenient parametrization in terms of the functions $V(t, x^c)$, $K^a(t, x^c)$, and $W^{\widehat{a}}{}_b(t, x^c)$ provides a unified description of all possible inertial and gravitational fields. We also include the classical electromagnetic field for completeness. In this general framework, we derive the Hermitian Dirac Hamiltonian (2.15). Starting with this master equation, we apply the Foldy-Wouthuysen transformation [54] and construct the Hamiltonian (3.11) in the FW representation for an *arbitrary spacetime geometry*. In this paper, we have confined ourselves to the purely Riemannian case of Einstein's general relativity theory, possible generalization to the non-Riemannian geometries will be analysed elsewhere. Making use of the FW Hamiltonian, we derive the operator equations of motion. In particular, we study the quantum-mechanical spin precession (3.15) and its semiclassical limit (3.18). One can apply these general results to compare the dynamics of a spinning particle in the inertial and gravitational fields, thus revisiting the validity of the equivalence principle [71]. The also derive the force operator and analyse the quantum dynamics of the particle under its action in Sec. III C. In the second part of the paper, we consider the motion of the classical particle with spin. In the framework of the Mathisson-Papapetrou theory, we obtained the dynamical equations (4.16), (4.17) and (4.26) which have a remarkably simple form of the motion of a relativistic particle under the action of the Lorentz force, with the Mathisson force included into the generalized gravitoelectric field (4.21). We also derived the equation (4.24) for the angular velocity of spin precession in the general gravitational field. It is satisfactory to see that our results further confirm the earlier conclusions [40, 41] and demonstrate that the classical spin dynamics is fully consistent with the semiclassical quantum dynamics of the Dirac fermion. Finally, the complete consistency of the quantum-mechanical and classical descriptions of spinning particles is also established using the Hamiltonian approach in Sec. IV B.

Among the important issues that remain still open, we would like to mention the need to carefully analyse the derivation of covariant equations of motion in the Dirac and in the Foldy-Wouthuysen representations. The crucial point in this study is to understand the definition of the position and spin operators in these two representations, in particular making use of the previous work on this subject in [72–82].

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