HAMILTON OPERATOR AND THE SEMICLASSICAL LIMIT FOR SCALAR PARTICLES IN AN ELECTROMAGNETIC FIELD

Alexander J. Silenko

Research Institute for Nuclear Problems of Belarusian State University, Minsk 220030, Belarus

Abstract

We successively apply the generalized Case-Foldy-Feshbach-Villars (CFFV) and the Foldy-Wouthuysen (FW) transformation to derive the Hamiltonian for relativistic scalar particles in an electromagnetic field. In contrast to the original transformation, the generalized CFFV transformation contains an arbitrary parameter and can be performed for massless particles, which allows solving the problem of massless particles in an electromagnetic field. We show that the form of the Hamiltonian in the FW representation is independent of the arbitrarily chosen parameter. Compared with the classical Hamiltonian for point particles, this Hamiltonian contains quantum terms characterizing the quadrupole coupling of moving particles to the electric field and the electric and mixed polarizabilities. We obtain the quantum mechanical and semiclassical equations of motion of massive and massless particles in an electromagnetic field.

Keywords: Klein-Gordon equation, Case-Foldy-Feshbach-Villars transformation, Foldy-Wouthuysen transformation, scalar particle, electromagnetic interaction

I. INTRODUCTION

The scalar (spinless) particles do not have their proper multipole moments. Therefore, studying their coupling to an external field gives an excellent opportunity to compare the conclusions of the classical and quantum theories. The additional terms in the quantum mechanical Hamiltonian describe the quantum interaction.

The original equation for spin-0 particles in an external field is the second-order Klein-Gordon (KG) equation, but if information about observable quantities is needed, then it is less convenient than the relativistic wave equation for the Hamiltonian or the first-order Duffin-Kemmer-Petiau (DKP) equation [1, 2, 3]. The DKP equation is successfully used in the case of particles with spins 0 and 1. Here, we transform the Hamiltonian to the diagonal form characterizing the Foldy-Wouthuysen (FW) representation [4]. Using this representation simplifies finding the eigenvalues and expectation values of operators and deriving the semiclassical equations of motion (see [5]).

Passing from a second-order equation to a first-order equation must be carefully done. The square root of both sides of an operator equation is often extracted in this case (see, e.g., [6]), but this may lead to inaccuracies [7]. Applying the Case-Foldy-Feshbach-Villars (CFFV) transformation to the equation for the Hamiltonian [8, 9, 10] and using the DKP equation for particles in an external field [11, 12, 13, 14, 15] are well-known and well-grounded methods.

The Hamiltonian for relativistic scalar point particles can be derived by the method proposed in [7]. based on successively applying the CFFV and FW transformations. The FW transformation for nonrelativistic scalar particles in an electromagnetic field was introduced in [8, 10, 16]. We here apply the generalized CFFV transformation, which can also be used in the massless particle case. We derive an equation for the Hamiltonian describing the coupling of massive and massless relativistic scalar particles to an electromagnetic field.

We use the system of units where $\hbar = c = 1$ but we explicitly introduce the constant $\hbar$ in some of the equations in Secs. 5 and 6.

II. PASSING TO THE FIRST-ORDER EQUATION FOR SCALAR PARTICLES

The original KG equation for scalar particles in an electromagnetic field has the form

$$\left[\left(i\frac{\partial}{\partial t}-e\varphi\right)^2-(\boldsymbol{p}-e\boldsymbol{A})^2-m^2\right]\psi=0, \tag{1}$$

where φ and $\boldsymbol{A}$ are the scalar and vector potentials of the electromagnetic field, $\boldsymbol{p}=-i\nabla$, e and m are the particle charge and mass, and ψ is a one-component wave function. This is a second-order equation. One of the main methods for studying the coupling of scalar particles to an external field consists in passing to a first-order equation in the time derivative. This can be done using the CFFV transformation proposed in [8, 9, 10]. Its result is a representation of the wave equation for scalar particles in Hamiltonian form.

We note that a similar transformation to a first-order equation is also done for spin-1 particles. Although the original relativistic second-order equations differ significantly for particles with spins 0 and 1, their transformations have several common features and can be done using similar methods. For spin-1 particles, such a transformation was done in [17]. The generalized Sakata-Taketani transformation for particles with an anomalous magnetic moment and an electric quadrupole moment was done in [18]. In all cases, the result of the transformations is an equation for an nondiagonal Hamiltonian acting on a bispinor wave function. The wave functions of the equation for the Hamiltonian obtained by the CFFV transformation can also be formally regarded as bispinors. The spinors for spinless particles are one-component, and the bispinor wave functions are two-component.

Although the equations for the nondiagonal Hamiltonians for particles with integer spin are formally similar to the Dirac equation, the form of the Hamiltonian and the wave function normalization differ significantly from those for spin-1/2 particles. At the same time, there is a certain analogy between the properties of the Hamiltonian and the wave functions for particles with spins 0 and 1. The wave functions for particles with spins 0, 1/2, and 1 (the bispinors) can be written in the general form

$$\Psi=\begin{pmatrix}\phi\\ \chi\end{pmatrix}, \tag{2}$$

where ϕ and χ are the higher- and lower-order spinors. The normalization of wave functions for particles with spins 0 and 1 is given by [18, 19]:

$$\int\Psi^\dagger\rho_3\Psi dV=1.$$

Here and hereafter, ρ_i $(i = 1, 2, 3)$ are the Pauli matrices, whose components act on the spinors ϕ and χ:

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

The Hamiltonian for particles with spins 0 and 1 is pseudo-Hermitian or, more precisely, β-pseudo-Hermitian (see [20] and the references therein) and non-Hermitian in the usual sense. It satisfies the relation [20] $(\beta^{-1} = \beta)$

$$\mathcal{H}^\dagger = \beta\mathcal{H}\beta,$$

which is equivalent to [18, 19]

$$\mathcal{H}^\ddagger \equiv \rho_3\mathcal{H}^\dagger\rho_3 = \mathcal{H}. \tag{4}$$

We define [19] the pseudoscalar product of two wave functions Ψ_1 and Ψ_2 as the integral [19]

$$< \Psi_1|\Psi_2 >= \int \Psi_1^\dagger\rho_3\Psi_2 dV, \tag{5}$$

Then, independently of the original representation, the pseudoscalar product remains unchanged under a pseudounitary transformation of the form $\Psi'_{1,2} = U\Psi_{1,2}$, whose operator has the property [19]:

$$U^\ddagger \equiv \rho_3 U^\dagger \rho_3 = U^{-1}. \tag{6}$$

For scalar particles, an nondiagonal Hamiltonian acting on a two-component wave function and obtained by transforming the second-order wave equation (the KG equation) was found by Case [8]. A method for obtaining the required transformation was described by Foldy [9]. The version of this method proposed by Feshbach and Villars [10] is now commonly used. A representation based on the nondiagonal Hamiltonian given above is therefore called the CFFV representation.

Using a pseudounitary transformation (such is the FW transformation), we can transform an nondiagonal Hamiltonian from the original CFFV representation (for spin-1 particles, the Sakata-Taketani representation) to the diagonal form (for spin-1 particles, to the block-diagonal form, i.e., to a form diagonal in two spinors). Such a transformation was done for relativistic free scalar particles and, in the nonrelativistic limit, for scalar particles in an electromagnetic field in [8, 10, 16].

Passing from the second-order KG equation to a first-order equation using the CFFV transformation is exact. The resulting equation determines an nondiagonal Hamiltonian acting on the two-component bispinor wave function (2).

A method for transforming to a first-order equation for free particles by passing to a two-component wave function Ψ was given in [9]. For particles in an external field, such a transformation, which is now commonly used, was done by Feshbach and Villars [10] (also see [19]). It consists in introducing wave functions satisfying the conditions [10, 19]:

$$\psi = \phi + \chi, \quad i\frac{\partial \psi}{\partial t} - e\varphi\psi = m(\phi - \chi). \tag{7}$$

In this case, the two-component wave function has the form

$$\Psi = \frac{1}{2}\begin{pmatrix} \psi + \frac{1}{m}\left[i\frac{\partial \psi}{\partial t} - e\varphi\psi\right] \\ \psi - \frac{1}{m}\left[i\frac{\partial \psi}{\partial t} - e\varphi\psi\right] \end{pmatrix}. \tag{8}$$

It was shown in [21] that the transformation of Eq. (1) into a first-order equation in the general case is done using formulas of form (7),(8), where the particle mass m can be replaced with any nonzero parameter. Because the CFFV transformation [8, 9, 10] can be used only for massive particles, it is interesting to consider the generalized CFFV transformation [21], which can also be used for massless particles. We study the generalized CFFV transformation in which the mass m can be replaced with an arbitrary nonzero real constant. The existence of an arbitrary parameter allows changing the form of the generalized CFFV transformation in a certain way. As a result, the form of the intermediate equation with an nondiagonal Hamiltonian (in the components of the wave function) in the generalized CFFV representation can also be changed. But, as is shown below, the final expression for the Hamiltonian in the FW representation is independent of the form of that intermediate equation.

The generalized transformation that we propose here can also be used for massless particles. This is very important because equations for massless particles cannot be obtained from equations for massive particles by passing to the limit as $m \to 0$ (see [22] and the references therein).

In the approach we use here, the wave functions ϕ and χ are determined by the equations

$$\psi = \phi + \chi, \quad \left(i\frac{\partial}{\partial t} - e\varphi\right)\psi = N(\phi - \chi), \tag{9}$$

where N is an arbitrary nonzero real parameter. If we multiply the last equation by $i\partial/(\partial t)-e\varphi$ then we can represent Eqs. (9) in matrix form:

$$i\frac{\partial\Psi}{\partial t}=\mathcal{H}\Psi,\quad \mathcal{H}=\rho_3\frac{\boldsymbol{\pi}^2+m^2+N^2}{2N}+e\varphi+i\rho_2\frac{\boldsymbol{\pi}^2+m^2-N^2}{2N}, \tag{10}$$

where $\boldsymbol{\pi}=\boldsymbol{p}-e\boldsymbol{A}=-i\nabla-e\boldsymbol{A}$ is the kinetic momentum operator and ρ_i are Pauli matrices (3), whose components act on the corresponding components of the wave function Ψ.

An equation equivalent to (10) was considered in [21]. Feshbach and Villars [10] considered a special form of this equation for $N=m$.

III. THE FOLDY-WOUTHUYSEN TRANSFORMATION FOR RELATIVISTIC PARTICLES IN AN EXTERNAL FIELD

The FW representation described in the classic paper [4] occupies a special place in relativistic quantum mechanics. In this representation, the relations between operators are completely analogous to the relations between the corresponding classical quantities. The operators in the FW representation have the same form as in the nonrelativistic quantum mechanics. It is very important that the coordinate operator $\boldsymbol{r}$ and momentum operator $\boldsymbol{p}=-i\nabla$ have a very simple form in this representation. Exactly the FW representation ensures the best possibility for obtaining the classical limit of relativistic quantum mechanics [4, 5, 23].

The quasidiagonal form (diagonal in two spinors) of the Hamiltonian is a result of the wave function transformation to a given representation (the FW transformation). The states with positive and negative total energy are separated in this case. The jitter problem (*Zitterbewegung*) does not arise in the FW representation. The FW transformation radically simplifies the process of passing to the semiclassical limit, which, in particular, does not require such a procedure as separating even parts of the operators.

We use the FW transformation in the one-particle approximation, where the radiation corrections are not calculated by field theory methods but are taken into account phenomenologically by including additional terms in the relativistic wave equations (similarly to the anomalous magnetic moment [24]). Naturally, the one-particle approximation can also be used in the relativistic particle case where the pair creation probability and the losses due to Bremsstrahlung can be neglected for a given coupling energy.

The Hamiltonian $\mathcal{H}$ can be decomposed into operators commuting and anticommuting with ρ_3:

$$\mathcal{H} = \rho_3\mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \rho_3\mathcal{M} = \mathcal{M}\rho_3, \quad \rho_3\mathcal{E} = \mathcal{E}\rho_3, \quad \rho_3\mathcal{O} = -\mathcal{O}\rho_3. \tag{11}$$

In the case under study, we have

$$\mathcal{M} = \frac{\boldsymbol{\pi}^2 + m^2 + N^2}{2N}, \quad \mathcal{E} = e\varphi, \quad \mathcal{O} = i\rho_2\frac{\boldsymbol{\pi}^2 + m^2 - N^2}{2N}. \tag{12}$$

It is very important that the equations determining the Hamiltonian for particles with spins 0, 1/2, and 1 have the same form (11). Therefore, the corresponding expressions for the operator U transforming the Hamiltonian to the block-diagonal form formally coincide. This allows using the FW transformation methods developed for particles with spin 1/2 [4, 5, 25, 26, 27, 28, 29] and spin 1 [30, 31] for the scalar particle. But we must take into account that not any arbitrary diagonalization of the Hamiltonian leads to the FW representation, as was first shown in [32]. As an example, we mention the Eriksen-Korlsrud transformation [33], which tranforms the Hamiltonian to the block-diagonal form. As proved in [34], it does not lead to the FW representation even in the free particle case.

There are several FW transformation methods for nonrelativistic spin-1/2 particles that allow calculating the relativistic corrections [4, 25, 27, 28, 29]. The classic method for such a transformation, proposed in [4] (also see [27]) for $\mathcal{M} = m$ consists in using the operator

$$U = e^{iS}, \quad S = -\frac{i}{2m}\rho_3\mathcal{O}. \tag{13}$$

The transformed Hamiltonian can be written as

$$\begin{gathered}\mathcal{H}' = \mathcal{H} + i[S,\mathcal{H}] + \frac{i^2}{2!}[S,[S,\mathcal{H}]] + \frac{i^3}{3!}[S,[S,[S,\mathcal{H}]]] + \dots \\ -\dot{S} - \frac{i}{2!}[S,\dot{S}] - \frac{i^2}{3!}[S,[S,\dot{S}]] - \dots,\end{gathered} \tag{14}$$

where $[\dots,\dots]$ is a commutator. As a result of transformation (14) the Hamiltonian is determined by the equation [4, 27]

$$\mathcal{H}' = \rho_3\epsilon + \mathcal{E}' + \mathcal{O}', \quad \rho_3\mathcal{E}' = \mathcal{E}'\rho_3, \quad \rho_3\mathcal{O}' = -\mathcal{O}'\rho_3, \tag{15}$$

where $\epsilon = \sqrt{m^2 + p^2}$ and the odd operator $\mathcal{O}'$ is now of the order $O(1/m)$. This transformation can be repeated multiply to obtain the required accuracy.

But for relativistic particles in an external field, it is generally rather complicated to pass to the FW representation. There are serious arguments [32] that the exact solution of this problem for Dirac particles in an arbitrary external field was obtained by Eriksen [25]. But the expression found in [25] contains square roots of matrix operators, which, as a rule, does not allow obtaining the Hamiltonian explicitly. For relativistic particles, it is also very difficult to represent it as a power series in the energy of coupling to the external field. Because this exact solution is very complicated, it was not used to perform the FW transformation for relativistic particles in an external field. Here, such a transformation is done using the method developed in [5] for spin-1/2 particles. This method allows finding the relativistic Hamiltonian in the form of a power series in the potentials of the external field and their derivatives. In some special cases, this method leads to the exact FW transformation [5].

The FW transformation was used for nonrelativistic spin-1 particles in an electromagnetic field in [18] and in the relativistic case in [30, 31]. It was used also for nonrelativistic scalar particles in an electromagnetic field in [8, 10, 16], and for relativistic particles in [7]. The nonrelativistic FW transformation was used for a system of two particles (a spin 0 boson and a spin-1/2 fermion) in [16].

Although the original relativistic wave equations differ significantly for particles of spins 0 and 1, their transformations to the FW representation have common features and are done using similar methods. First, a common feature is that a preliminary transformation taking the original equations to the Hamiltonian form is needed. These are the CFFV transformation [8, 9, 10] for scalar particles and the Sakata-Taketani transformation [17] for spin-1 particles. The generalized Sakata-Taketani transformation for particles with an anomalous magnetic moment and an electric quadrupole moment was done in [18]. In all cases, the result is an equation for the nondiagonal Hamiltonian acting on a bispinor wave function.

The exact FW transformation can be done under the commutation conditions

$$[\mathcal{M},\mathcal{O}]=0, \quad [\mathcal{E},\mathcal{O}]=0, \tag{16}$$

and in the case of a stationary external field. In this case, the Hamiltonian $\mathcal{H}$ is transformed to the block-diagonal form using the operator [5, 31]

$$U=\frac{\epsilon+\mathcal{M}+\rho_3\mathcal{O}}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}}, \quad U^{-1}=\frac{\epsilon+\mathcal{M}-\rho_3\mathcal{O}}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}}, \quad \epsilon=\sqrt{\mathcal{M}^2+\mathcal{O}^2}. \tag{17}$$

The transformed Hamiltonian has the form [5, 31]

$$\mathcal{H}' = \rho_3\epsilon + \mathcal{E}. \tag{18}$$

In the general case determined by formulas (11) and (12), the external field is nonstationary, and the operator $\mathcal{O}$ commutes with $\mathcal{M}$ but may not commute with $\mathcal{E}$. We calculate in the weak-field approximation and assume that the coupling energy is small compared with the total energy, which includes the rest energy mc^2 and is approximately equal to ϵ. If the weak-field approximation is used, then the small dimensionless parameters in which we expand are the ratios of the terms in the operator of the particle coupling to an external field (which are proportional to the first powers of the field potentials and their space and time derivatives) to the total particle energy. The complete solution of the problem gives an expression for the Hamiltonian in the FW representation as a power series in the potentials of the external field and their derivatives.

The method used here and developed in [5, 31] consists in the following. First, we perform a pseudounitary transformation with operator (17) (see [5, 31]). After this transformation, the Hamiltonian $\mathcal{H}'$ still contains odd terms proportional to the potential derivatives and can be written as

$$\mathcal{H}' = \rho_3\epsilon + \mathcal{E}' + \mathcal{O}', \quad \rho_3\mathcal{E}' = \mathcal{E}'\rho_3, \quad \rho_3\mathcal{O}' = -\mathcal{O}'\rho_3, \tag{19}$$

where (see [5])

$$\begin{gathered}
\mathcal{E}' = \mathcal{E} - \frac{1}{4}\left[\frac{\epsilon+\mathcal{M}}{\sqrt{\epsilon(\epsilon+\mathcal{M})}}, \left[\frac{\epsilon+\mathcal{M}}{\sqrt{\epsilon(\epsilon+\mathcal{M})}}, \mathcal{F}\right]\right] \\
+\frac{1}{4}\left[\frac{\beta\mathcal{O}}{\sqrt{\epsilon(\epsilon+\mathcal{M})}}, \left[\frac{\beta\mathcal{O}}{\sqrt{\epsilon(\epsilon+\mathcal{M})}}, \mathcal{F}\right]\right], \\
\mathcal{O}' = \frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}}\mathcal{F}\frac{\epsilon+\mathcal{M}}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}} - \frac{\epsilon+\mathcal{M}}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}}\mathcal{F}\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+\mathcal{M})}}, \\
\mathcal{F} = \mathcal{E} - i\frac{\partial}{\partial t},
\end{gathered} \tag{20}$$

and ϵ is determined by formula (17). If the weak-field approximation is used, then the odd operator $\mathcal{O}'$ is small compared with both ϵ and the original Hamiltonian $\mathcal{H}$. The usual scheme of the nonrelativistic FW transformation [4, 5, 27, 31] can therefore be used at the second stage. Such a transformation is done using the operator

$$U' = \exp(iS'), \quad S' = -\frac{i}{4}\rho_3\left\{\mathcal{O}', \frac{1}{\epsilon}\right\} = -\frac{i}{4}\left[\frac{\rho_3}{\epsilon}, \mathcal{O}'\right], \tag{21}$$

where $\{\ldots,\ldots\}$ is an anticommutator. The further calculations are the same as in the case of particles with spins 1/2 and 1 (see [5, 27, 31]). In contrast to [27], the particle mass in this case must be replaced with the operator ϵ noncommuting with the operators $\mathcal{E}'$ and $\mathcal{O}'$. If we consider only the leading corrections, which are proportional to $\mathcal{O}'^2$, i.e., to the second powers of the field potentials and their space and time derivatives, then the transformed Hamiltonian becomes [5]

$$\mathcal{H}_{FW} = \mathcal{H}'' = \rho_3\epsilon + \mathcal{E}' + \frac{\rho_3}{4}\left\{\frac{1}{\epsilon}, \mathcal{O}'^2\right\}. \tag{22}$$

The transformation with operator (21) can be repeated multiply (S' is replaced with S'', S''' and so on) to obtain the required accuracy.

IV. THE FOLDY-WOUTHUYSEN TRANSFORMATION FOR SCALAR PARTICLES

We perform the FW transformation for relativistic scalar particles in an electromagnetic field using the method described above. We assume that the coupling energy is small compared with the total energy including the rest energy. For the Hamiltonian determined by formulas (11) and (12), we have

$$\epsilon = \sqrt{m^2 + \boldsymbol{\pi}^2}, \tag{23}$$

and the pseudounitary operator of transformation (17) in this case can be reduced to the form

$$U = \frac{\epsilon + N + \rho_1(\epsilon - N)}{2\sqrt{\epsilon N}}. \tag{24}$$

As a result of the transformation of the original Hamiltonian determined by formulas (11) and (12) using operator (24), the transformed Hamiltonian $\mathcal{H}'$ also contains odd terms and has form (19), where

$$\mathcal{E}' = i\frac{\partial}{\partial t} + \frac{1}{2}\left(\sqrt{\epsilon}\mathcal{F}\frac{1}{\sqrt{\epsilon}} + \frac{1}{\sqrt{\epsilon}}\mathcal{F}\sqrt{\epsilon}\right), \quad \mathcal{O}' = \frac{\rho_1}{2}\left(\sqrt{\epsilon}\mathcal{F}\frac{1}{\sqrt{\epsilon}} - \frac{1}{\sqrt{\epsilon}}\mathcal{F}\sqrt{\epsilon}\right), \tag{25}$$

and ϵ is determined by formula (23). The commutation relations can be used to transform formulas (25) as

$$\mathcal{E}' = \mathcal{E} + \frac{1}{2\sqrt{\epsilon}}\left[\sqrt{\epsilon}, [\sqrt{\epsilon}, \mathcal{F}]\right]\frac{1}{\sqrt{\epsilon}}, \quad \mathcal{O}' = \rho_1\frac{1}{2\sqrt{\epsilon}}[\epsilon, \mathcal{F}]\frac{1}{\sqrt{\epsilon}}. \tag{26}$$

Formulas (23), (25), and (26) are exact for an arbitrary operator $\mathcal{E}$ and are independent of N. This means that the Hamiltonian in the FW representation, obtained at the next stage of transformations, is also independent of N. The CFFV representation does not have such a property. Thus, the successive generalized CFFV and FW transformations again demonstrate the special role of the FW representation in particle physics, this time with an example of spin-0 particles.

Approximately (see the general formulas for commutators in [5]),

$$[\epsilon,\mathcal{F}]=\frac{1}{4}\left\{\frac{1}{\epsilon},[\boldsymbol{\pi}^2,\mathcal{F}]\right\},\qquad \left[\sqrt{\epsilon},[\sqrt{\epsilon},\mathcal{F}]\right]=\frac{1}{32}\left\{\frac{1}{\epsilon^3},[\boldsymbol{\pi}^2,[\boldsymbol{\pi}^2,\mathcal{F}]]\right\}.$$

The Hamiltonian in the FW representation, which can be obtained by approximate formula (22), is equal to

$$\mathcal{H}_{FW}=\rho_3\epsilon+\mathcal{E}+\frac{1}{64}\left\{\frac{1}{\epsilon^4},[\boldsymbol{\pi}^2,[\boldsymbol{\pi}^2,\mathcal{F}]]\right\}+\frac{\rho_3}{64}\left\{\frac{1}{\epsilon^5},\left([\boldsymbol{\pi}^2,\mathcal{F}]\right)^2\right\}. \tag{27}$$

The above calculations are general and formulas (23)-(27) hold in the case of the coupling of scalar particles to any external field whose Hamiltonian is determined by (11) and (12) with arbitrary $\mathcal{E}$ and $\boldsymbol{\pi}$. In the case of electromagnetic interaction described by formulas (11) and (12), we have

$$\begin{gathered}\left[\boldsymbol{\pi}^2,\mathcal{F}\right]=2ie\boldsymbol{\pi}\cdot\boldsymbol{E},\qquad \left[\boldsymbol{\pi}^2,\left[\boldsymbol{\pi}^2,\mathcal{F}\right]\right]=4e(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})-4e^2\boldsymbol{\pi}\cdot(\boldsymbol{E}\times\boldsymbol{H}),\\ (\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\equiv\pi_i\pi_j\frac{\partial E_j}{\partial x_i}.\end{gathered} \tag{28}$$

As a result, we obtain the final approximate expression for the Hamiltonian in the FW representation characterizing the coupling of scalar point particles to an electromagnetic field:

$$\mathcal{H}_{FW}=\rho_3\epsilon+e\varphi+\frac{e}{8\epsilon^4}(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})-\frac{e^2}{8\epsilon^4}\boldsymbol{\pi}\cdot(\boldsymbol{E}\times\boldsymbol{H})-\rho_3\frac{e^2}{8\epsilon^5}(\boldsymbol{\pi}\cdot\boldsymbol{E})^2. \tag{29}$$

We note that the successive generalized CFFV and FW transformations allow deriving a Hamiltonian describing both massive and massless particles, and this solves the problem of massless particles in an electromagnetic field. As follows from (7) and (8), the original transformation [8, 9, 10] cannot be done for $m=0$. If the original method [8, 9, 10] is used, then the massless particle case cannot be regarded as the limit case (as $m\to 0$) for massive particles, because the mass is a multiplier in this approach and it is hence useless to find the limit as $m\to 0$. Precisely the same situation arises if we try to use this limit to describe the massless particles by the DKP equation (see [22]).

In formulas (28) and (29), we do not take the order of operators into account, bearing in mind the subsequent semiclassical approximation. In this approach, the condition that the average commutators of the operators of dynamical variables (coordinates and momentum) are small compared with the average products of these operators must be satisfied. The corresponding noncommuting operators in the quantum mechanical expressions can be rearranged in this case.

V. COMPARISON OF THE PARTICLE PROPERTIES IN CLASSICAL AND QUANTUM ELECTRODYNAMICS

The semiclassical transition can be done analogously to [5]. The above condition that the average commutators of the operators of coordinates and momentum are small compared with the average products of these operators is satisfied automatically if the characteristic length of the domain of the external field inhomogeneity is significantly greater than the de Broglie wavelength of the particle: $l \gg \hbar/p$. In this case, the semiclassical transition is achieved by a trivial replacement of the operators in the quantum mechanical equations for the higher-order spinor with the corresponding classical quantities. The semiclassical Hamiltonian thus obtained is given by

$$\mathcal{H}_s = \epsilon + e\varphi + \frac{e}{8\epsilon^4}(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E}) - \frac{e^2}{8\epsilon^4}\boldsymbol{\pi}\cdot(\boldsymbol{E}\times\boldsymbol{H}) - \frac{e^2}{8\epsilon^5}(\boldsymbol{\pi}\cdot\boldsymbol{E})^2. \tag{30}$$

The corresponding classical Hamiltonian contains only the first two terms:

$$\mathcal{H}_c = \epsilon + e\varphi. \tag{31}$$

Although we deal with point particles, the last three terms in semiclassical Hamiltonian (29) describe properties typical of classical composite particles. But the Hamiltonian does not contain any term characterizing the contact (Darwinian) interaction of rest particles. Moreover, all quantum corrections become zero for rest particles. This property agrees with the results obtained in [35]. The derivation of the Hamiltonian presented there for particles with arbitrary spin and the subsequent FW transformation for nonrelativistic particles showed that the appearance of the static contact (Darwinian) interaction is related to the particle spin. The absence of such an interaction for spin-0 particles is manifested by the term characterizing the contact interaction [35] becoming infinite for $S = 0$.

Restoring the constant $\hbar$ in Eq. (29) shows that the last three terms are proportional to $\hbar^2$:

$$\mathcal{H}_s = \epsilon + e\varphi + \frac{e\hbar^2}{8\epsilon^4}(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E}) - \frac{e^2\hbar^2}{8\epsilon^4}\boldsymbol{\pi}\cdot(\boldsymbol{E}\times\boldsymbol{H}) - \frac{e^2\hbar^2}{8\epsilon^5}(\boldsymbol{\pi}\cdot\boldsymbol{E})^2. \tag{32}$$

Moving particles have several nonclassical properties. The third term in (29) describes the quadrupole coupling to the electric field. A similar term is contained in the Hamiltonian for spin-1/2 particles [5, 26]. The last term characterizes the electric polarizability of moving particles, which is also nonzero for spin-1/2 point particles [26], and the preceding term characterizes the mixed polarizability. The induced electric dipole moment is equal to

$$\boldsymbol{d} = \frac{\partial\mathcal{H}_s}{\partial\boldsymbol{E}} = \frac{e^2\hbar^2}{8\epsilon^4}\boldsymbol{\pi}\times\boldsymbol{H} - \frac{e^2\hbar^2}{4\epsilon^5}\boldsymbol{\pi}(\boldsymbol{\pi}\cdot\boldsymbol{E}). \tag{33}$$

The induced magnetic dipole moment is determined by the formula

$$\boldsymbol{\mu} = \frac{\partial\mathcal{H}_s}{\partial\boldsymbol{H}} = -\frac{e^2\hbar^2}{8\epsilon^4}\boldsymbol{\pi}\times\boldsymbol{E}. \tag{34}$$

Hence, the induced electric dipole moment is proportional to the magnetic field strength, and the induced magnetic dipole moment is proportional to the electric field strength. These are quantum properties. The quantum mechanical equation of particle motion has the form

$$\frac{d\boldsymbol{\pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW},\boldsymbol{\pi}] - e\frac{\partial\boldsymbol{A}}{\partial t}. \tag{35}$$

In the case under study, its semiclassical limit is determined by the formula

$$\begin{gathered}\frac{d\boldsymbol{\pi}}{dt} = e\boldsymbol{E} + \frac{e}{\epsilon}(\boldsymbol{\pi}\times\boldsymbol{H}) + \frac{e^2\hbar^2}{8\epsilon^4}\left[(\boldsymbol{\pi}\cdot\nabla)(\boldsymbol{E}\times\boldsymbol{H}) - (\boldsymbol{H}\times\nabla)(\boldsymbol{\pi}\cdot\boldsymbol{E})\right] \\ + \frac{e^3\hbar^2}{8\epsilon^4}\left[H^2\boldsymbol{E} - \boldsymbol{H}(\boldsymbol{E}\cdot\boldsymbol{H})\right] - \frac{e^3\hbar^2}{4\epsilon^5}(\boldsymbol{\pi}\cdot\boldsymbol{E})(\boldsymbol{E}\times\boldsymbol{H}).\end{gathered} \tag{36}$$

The quantum corrections to the classical expression determining the Lorentz force are very small.

VI. THE DUFFIN-KEMMER-PETIAU EQUATION FOR PARTICLES IN AN ELECTROMAGNETIC FIELD AND THE PROBLEM OF MASSLESS PARTICLES

The DKP equation [1, 2, 3] for particles with spins 0 and 1 is an analogue of the Dirac equation. For free scalar particles, it has the form

$$(\beta^\mu\partial_\mu - m)\Phi = 0, \tag{37}$$

where

$$\beta^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \beta^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\beta^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{38}$$

The covariant generalization of this equation for spin 0 particles in an electromagnetic field is [11, 12, 13]

$$(\beta^\mu D_\mu - m)\Phi = 0, \quad D_\mu = \partial_\mu + ieA_\mu, \tag{39}$$

where $A_\mu = (\varphi, -\boldsymbol{A})$ A) is the four-dimensional potential of the electromagnetic field. We write Eq. (39) explicitly as

$$\begin{gathered} -D_0\Phi_5 = m\Phi_1, \quad D_1\Phi_5 = m\Phi_2, \quad D_2\Phi_5 = m\Phi_3, \quad D_3\Phi_5 = m\Phi_4, \\ D_0\Phi_1 + D_1\Phi_2 + D_2\Phi_3 + D_3\Phi_4 = m\Phi_5. \end{gathered} \tag{40}$$

The equivalence of the DKP and KG equations is proved not only for the electromagnetic interactions [11, 12, 13] but also for some others [12, 13, 15]. Eliminating the components Φ_2, Φ_3, and Φ_4 from (40), we obtain the relations

$$-D_0\Phi_5 = m\Phi_1, \quad D_0\Phi_1 = \left(m - \frac{D_1^2 + D_2^2 + D_3^2}{m}\right)\Phi_5. \tag{41}$$

The transformation $\Phi_1 = \phi - \chi,\ \Phi_5 = -i(\phi + \chi)$ allows obtaining the equation

$$iD_0\Psi = \left[\rho_3\left(m - \frac{D_1^2 + D_2^2 + D_3^2}{2m}\right) - i\rho_2\frac{D_1^2 + D_2^2 + D_3^2}{2m}\right]\Psi, \quad \Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{42}$$

which is equivalent to (10) under the condition that $N = m$. Eliminating Φ_1 from (41), we obtain the original Eq. (1).

But DKP equation (39) is inapplicable to massless particles even in the limit as $m \to 0$ [22], while the method proposed here based on the generalized CFFV transformation can also be used in that case. A modification of the DKP equation that allows describing free massless particles was found in [22]. The relativistic wave equation for the Hamiltonian in the FW representation gives the complete quantum mechanical description of the coupling of massless particles to an electromagnetic field. Naturally, this description does not take the quantum field effects into account. In accordance with formula (29) for the Hamiltonian, this equation for massless particles has the form

$$i\hbar\frac{\partial\Psi_{FW}}{\partial t}=\left[\rho_3\epsilon+e\varphi+\frac{e\hbar^2}{8c^2\epsilon^2}\cdot\frac{\partial E_x}{\partial x}-\frac{e^2\hbar^2}{8c\epsilon^3}(\boldsymbol{E}\times\boldsymbol{H})_x-\rho_3\frac{e^2\hbar^2}{8c^2\epsilon^3}E_x^2\right]\Psi_{FW}, \tag{43}$$

where $\epsilon=c\sqrt{\boldsymbol{\pi}^2}$ and the direction of the particle motion $\boldsymbol{l}=c\boldsymbol{\pi}/\epsilon$ is taken as the x axis. In the FW representation, we can use only the higher-order spinor because the lower-order spinor characterizing the states with negative total energy is equal to zero for real particles. In this case, the relativistic wave equation becomes

$$i\hbar\frac{\partial\phi_{FW}}{\partial t}=\left[\epsilon+e\varphi+\frac{e\hbar^2}{8c^2\epsilon^2}\cdot\frac{\partial E_x}{\partial x}-\frac{e^2\hbar^2}{8c\epsilon^3}(\boldsymbol{E}\times\boldsymbol{H})_x-\frac{e^2\hbar^2}{8c^2\epsilon^3}E_x^2\right]\phi_{FW}. \tag{44}$$

Equations (29), (43), and (44) (just as the original KG equation) are equivalent to the DKP equation, but their distinguishing feature is that the quantum corrections to the classical Hamiltonian are represented explicitly. These corrections are relativistic and are zero in the static limit. Such corrections also appear in the quantum mechanical Hamiltonian for spin-1/2 particles [5, 26].

VII. CONCLUSIONS

We have transformed the original KG equation for scalar particles in an electromagnetic field to the Hamiltonian form by successively using the generalized CFFV and FW transformations. We derived relativistic formulas for the Hamiltonian and the semiclassical Hamiltonian. In contrast to the original transformation, the generalized CFFV transformation contains an arbitrary parameter and can hence be used for massless particles. We showed that the form of the Hamiltonian in the FW representation is independent of the arbitrarily chosen parameter. We solved the problem of massless particles in an electromagnetic field. Compared with the classical Hamiltonian for point particles, the Hamiltonian operator and the semiclassical Hamiltonian contain quantum terms characterizing the quadrupole coupling of moving particles to the electric field and the electric and mixed polarizabilities.

These terms are relativistic and are zero in the static limit. Such quantum corrections to the classical Hamiltonian also appear in the case of spin-1/2 particles. We obtained quantum mechanical and semiclassical equations of motion for massive and massless particles in an electromagnetic field and compared the obtained equations with the DKP equation.

Acknowledgments. This work is supported by the Belarusian Foundation for Basic Research.

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