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Radiative instability of a relativistic electron beam moving in a photonic crystal

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Abstract

The radiative instability of a beam moving in a photonic crystal of finite dimensions is studied. The dispersion equation is obtained. The law $\Gamma \sim \rho^{1/(s+3)}$ is shown to be valid and caused by the mixing of the electromagnetic field modes in the finite volume due to the periodic disturbance from the photonic crystal.

Numerous works analyze the generation of induced radiation by electron beams moving in spatially periodic media [1,2,3,4,5]. Thoroughly studied are the processes of wave generation in a one-dimensional case, when the electron beam moves along the axis of a corrugated waveguide (traveling-wave tube (TWT), backward-wave tube (BWT)) or along the axis of the undulator (volume free electron lasers, ubitrons). It was found out that under the action of radiation, the electron beam with a uniform density distribution becomes spatially modulated, i.e., the radiative instability of a beam emerges. The increment of the electron beam instability is a most important quantity characterizing the generation capability of the beam. The analysis of all mechanisms of induced radiation generation by relativistic beams in the Compton regime showed that the increment of beam instability Γ for a cold beam (i.e., a beam where all the electrons have the same longitudinal velocity $\vec{u}$) follows the law $\Gamma \sim \rho^{1/3}$, where ρ - is the beam density [2,3].

In 1984 it was shown [6] that induced X-ray radiation produced by the electron beam passing through the crystal under the conditions providing the coincidence of roots of the dispersion equation, which describes the relation between the wave vector $\vec{k}$ and the photon frequency ω in the crystal, leads to the appearance of a new law for the increment of the beam instability: $\Gamma \sim \rho^{1/(s+3)}$, where s is the number of additional waves appearing due to diffraction of emitted X-ray quanta in the crystal. This law of instability leads to a significant reduction of the laser generation threshold for X-ray radiation in the crystal (according to this law, the generation threshold for a LiH crystal can be achieved when the beam current density is 10^8 $\mathrm{A/cm^2}$ in contrast to 10^{13} $\mathrm{A/cm^2}$ as required according to the conventional law $\rho^{1/3}$.

It was later shown in [7,8] that the above described law is valuid for all wavelength ranges of induced radiation produced by electrons in spatially periodic media (diffraction gratings) and for various types of nonlinear interaction of waves in both natural and artificial (or frequently called "photonic") crystals. The results obtained in these works enable design and development of a new type of Free Electron Lasers called the Volume free Electron Lasers (VFEL) [9,10].

Theoretical description of the generation processes in a photonic crystal placed inside the resonator was given in [9,12,13]. The first and most important step in describing the generation process in VFELs (FELs and so on) is the analysis of the problem of the electron beam instability in the resonator. The theoretical study of the instability of electron beams moving in natural and artificial (photonic) crystals was carried out for the ideal case of an infinite medium (see the review in [9] and [6,10,11,12,13]). The question arising in this regard is how the finite dimensions of the photonic crystal placed inside the resonator affect the law of electron beam instability. It is known, for example, that the discrete structure of the modes in waveguides and resonators is crucial for effective generation in the microwave range [2,3,4,5].

In present paper the radiative instability of a beam moving in a photonic crystal is studied. The dispersion equation describing instability in this case is obtained. It is shown that the law $\Gamma \sim \rho^{1/(s+3)}$ is also valid and caused by the mixing of the electromagnetic field modes in the finite volume due to the periodic disturbance from the photonic crystal.

The system of equations describing generation of induced radiation in photonic (and natural) crystals can be obtained from Maxwell equations:

$$
\begin{gathered}
rot\vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{j},\ rot\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}, \\
div\vec{D} = 4\pi\rho,\ \frac{\partial \rho}{\partial t} + div\vec{j} = 0,
\end{gathered}
\tag{1}
$$

where $\vec{E}$ and $\vec{H}$ are the strength of the electric and the magnetic field, respectively; $\vec{j}$ ρ are the current and charge densities; $D_i\left(\vec{r}, t'\right) = \int \varepsilon_{il}\left(\vec{r}, t-t'\right) E_l\left(\vec{r}, t'\right) dt'$ or $D_i\left(\vec{r}, \omega\right) = \varepsilon_{il}\left(\vec{r}, \omega\right) E_l\left(\vec{r}, \omega\right)$, where indices $i, l = 1, 2, 3$ correspond to x, y, z; $\varepsilon_{il}\left(\vec{r}, \omega\right)$ is the dielectric permittivity tensor of the photonic crystal.

The current and charge densities are defined as:

$$\vec{j}\,(\vec{r},t) = e\sum_{\alpha}\vec{v}_{\alpha}\delta(\vec{r}-\vec{r}_{\alpha}(t)),\;\; \rho(\vec{r},t) = e\sum_{\alpha}\delta(\vec{r}-\vec{r}_{\alpha}(t)), \tag{2}$$

where e is the electron charge, $\vec{v}_{\alpha}$ is the velocity of the electron with number α in the electron beam,

$$\frac{d\vec{v}_{\alpha}}{dt} = \frac{e}{m\gamma_{\alpha}}\left\{\vec{E}\left(\vec{r}_{\alpha}\left(t\right),t\right)+\frac{1}{c}\left[\vec{v}_{\alpha}\left(t\right)\times\vec{H}\left(\vec{r}_{\alpha}\left(t\right),t\right)\right]-\frac{\vec{v}_{\alpha}}{c^{2}}\left(\vec{v}_{\alpha}\left(t\right)\vec{E}\left(\vec{r}_{\alpha}\left(t\right),t\right)\right)\right\}, \tag{3}$$

where $\gamma_{\alpha} = \left(1-\frac{v_{\alpha}^{2}}{c^{2}}\right)^{-\frac{1}{2}}$ is the Lorentz factor, $\vec{E}\left(\vec{r}_{\alpha}\left(t\right),t\right)$ ($\vec{H}\left(\vec{r}_{\alpha}\left(t\right),t\right)$) is the electric (magnetic) field strength at point $\vec{r}_{\alpha}$, where the electron with number α is located. Note that equation (3) can be written as follows [14]:

$$\frac{d\vec{p}_{\alpha}}{dt} = m\frac{d\gamma_{\alpha}v_{\alpha}}{dt} = e\left\{\vec{E}\left(\vec{r}_{\alpha}\left(t\right),t\right)+\frac{1}{c}\left[\vec{v}_{\alpha}\left(t\right)\times\vec{H}\left(\vec{r}_{\alpha}\left(t\right),t\right)\right]\right\}, \tag{4}$$

where p_{α} is the particle momentum.

From equations (2) one can obtain

$$-\Delta\vec{E}+\vec{\nabla}\left(\vec{\nabla}\vec{E}\right)+\frac{1}{c^{2}}\frac{\partial^{2}\vec{D}}{\partial t^{2}} = -\frac{4\pi}{c^{2}}\frac{\partial\vec{j}}{\partial t}. \tag{5}$$

The dielectric permittivity tensor can be presented in the form $\hat{\varepsilon}\left(\vec{r}\right) = 1+\hat{\chi}\left(\vec{r}\right)$, where $\hat{\chi}\left(\vec{r}\right)$ is the susceptibility.

For $\hat{\chi} << 1$, equation (5) can be rewritten as

$$\Delta\vec{E}\left(\vec{r},t\right)-\frac{1}{c^{2}}\frac{\partial^{2}}{\partial t^{2}}\int\hat{\varepsilon}\left(\vec{r},t-t'\right)\vec{E}\left(\vec{r},t'\right)dt' = 4\pi\left(\frac{1}{c^{2}}\frac{\partial\vec{j}\left(\vec{r},t\right)}{\partial t}+\vec{\nabla}\rho\left(\vec{r},t\right)\right). \tag{6}$$

In the general case, the susceptibility of the photonic crystal reads $\hat{\chi}\left(\vec{r}\right) = \sum_{i}\hat{\chi}_{cell}\left(\vec{r}-\vec{r}_{i}\right)$, where $\hat{\chi}_{cell}\left(\vec{r}-\vec{r}_{i}\right)$ is the susceptibility of the crystal unit cell. The susceptibility of an infinite perfect crystal $\hat{\chi}\left(\vec{r}\right)$ can be expanded into the Fourier series as follows: $\hat{\chi}\left(\vec{r}\right) = \sum_{\vec{\tau}}\hat{\chi}_{\vec{\tau}}e^{i\vec{\tau}\vec{r}}$, where $\vec{\tau}$ is the reciprocal lattice vector of the crystal.

To be more specific, let us consider in details a practically important case when a photonic crystal is placed inside a smooth waveguide of rectangular cross-section.

The eigenfunctions and the eigenvalues of such a waveguide are well-studied [15,16].

Suppose the z-axis to be directed along the waveguide axis. Make the Fourier transform of (5) over time and longitudinal coordinate z . Expanding thus obtained equation for the field $\vec{E}\left(\vec{r}_{\perp}, k_z, \omega\right)$ over a full set of vector eigenfunctions of a rectangular waveguide $\vec{Y}_{mn}^{\lambda}\left(\vec{r}_{\perp}, k_z\right)$ (where $m, n = 1, 2, 3....$, while λ describes the type of the wave [17], one can obtain for the field $\vec{E}$ the equality

$$\vec{E}\left(\vec{r}_{\perp}, k_z, \omega\right) = \sum_{mn\lambda} C_{mn}^{\lambda}\left(k_z, \omega\right) \vec{Y}_{mn}^{\lambda}\left(\vec{r}_{\perp}, k_z\right). \tag{7}$$

As a result, the following equation can be written

$$\begin{aligned} &\left[\left(k_z^2 + \kappa_{mn\lambda}^2\right) - \tfrac{\omega^2}{c^2}\right] C_{mn}^{\lambda}\left(k_z, \omega\right) - \\ &-\tfrac{\omega^2}{c^2}\tfrac{1}{2\pi}\sum_{m'n'\lambda'}\int \vec{Y}_{mn}^{\lambda *}\left(\vec{r}_{\perp}, k_z\right)\hat{\chi}\left(\vec{r}\right)\vec{Y}_{m'n'}^{\lambda'}\left(\vec{r}_{\perp}, k_z'\right)e^{-i\left(k_z - k_z'\right)z}\, d^2 r_{\perp} C_{m'n'}^{\lambda'}\left(k_z', \omega\right) dk_z' dz = \\ &= \tfrac{4\pi i\omega}{c^2}\int \vec{Y}_{mn}^{\lambda *}\left(\vec{r}_{\perp}, k_z\right)\left\{\vec{j}\left(\vec{r}_{\perp}, z, \omega\right) + \tfrac{c^2}{\omega^2}\vec{\nabla}\left(\vec{\nabla}\vec{j}\left(\vec{r}_{\perp}, z, \omega\right)\right)\right\} e^{-ik_z z} d^2 r_{\perp} dz \end{aligned} \tag{8}$$

where $\kappa_{mn\lambda}^2 = k_{xm\lambda}^2 + k_{yn\lambda}^2$.

The beam current and density appearing on the right-hand side of (8) are complicated functions of the field $\vec{E}$. To study the problem of the system instability, it is sufficient to consider the system in the approximation linear over perturbation, i.e., one can expand the expressions for $\vec{j}$ and ρ over the field amplitude $\vec{E}$ and abridge oneself with the linear approximation.

As a result, a closed system of equations comes out. For further consideration, one should obtain the expressions for the corrections $\delta\vec{j}$ and $\delta\rho$ due to beam perturbation by the field. Considering the Fourier transforms of the current density and the beam charge $\vec{j}\left(\vec{k}, \omega\right)$ and $\rho\left(\vec{k}, \omega\right)$, one can obtain from (2) that

$$\delta\vec{j}\left(\vec{k}, \omega\right) = e\sum_{\alpha=1}^{N} e^{-i\vec{k}\vec{r}_{\alpha 0}}\left\{\delta\vec{v}_{\alpha}\left(\omega - \vec{k}\vec{u}_{\alpha}\right) + \vec{u}_{\alpha}\frac{\vec{k}\delta\vec{v}_{\alpha}\left(\omega - \vec{k}\vec{u}_{\alpha}\right)}{\omega - \vec{k}\vec{u}_{\alpha}}\right\}, \tag{9}$$

where $\vec{r}_{\alpha_0}$ is the original coordinate of the electron, $\vec{u}_{\alpha}$ is the unperturbed velocity of the electron.

For simplicity, let us consider a cold beam, for which $\vec{u}_{\alpha} \approx \vec{u}$, where $\vec{u}$ is the mean velocity of the beam. The general case of a hot beam is obtained by averaging $\delta\vec{j}\left(\vec{k}, \omega\right)$ over the velocity $\vec{u}_{\alpha}$ distribution in the beam.

According to (3), the velocity $\delta\vec{v}_\alpha$ is determined by the field $\vec{E}\left(\vec{r}_\alpha,\omega\right)$ taken at the electron location point $\vec{r}_\alpha$. The Fourier transform of the field $\vec{E}\left(\vec{r}_\alpha,\omega\right)$ has a form

$$\vec{E}\left(\vec{r}_\alpha,\omega\right)=\frac{1}{\left(2\pi\right)^3}\int\vec{E}\left(\vec{k}',\omega\right)e^{i\vec{k}'\vec{r}_\alpha}d^3k'.$$

As a result, the formula for $\delta\vec{j}\left(\vec{k},\omega\right)$ includes the sum $\sum\limits_\alpha e^{-i\left(\vec{k}-\vec{k}'\right)\vec{r}_\alpha}$ over the particle distribution in the beam. Suppose that the electrons in an unperturbed beam are uniformly distributed over the area occupied by the beam. Therefore

$$\sum_\alpha e^{-i\left(\vec{k}-\vec{k}'\right)\vec{r}_\alpha}=\left(2\pi\right)^3\rho_0\,\delta\left(\vec{k}-\vec{k}'\right),$$

where ρ_0 is the beam density (the number of electrons per 1 cm^3).

As a result, the following expression for $\delta\vec{j}\left(\vec{k},\omega\right)$ can be obtained [18,19]:

$$\delta\vec{j}\left(\vec{k},\omega\right)=\frac{i\vec{u}e^2\rho\left(k^2-\frac{\omega^2}{c^2}\right)}{\left(\omega-\vec{k}\vec{u}\right)^2m\gamma\omega}\vec{u}\vec{E}\left(\vec{k},\omega\right). \tag{10}$$

Using the continuity equation, one immediately obtains the expression for $\rho\left(\vec{k},\omega\right)$. Expression (10), the inverse Fourier transform of $\vec{E}\left(\vec{k},\omega\right)$, and the expansion (7) enable writing the system of equations (8) as follows:

$$\begin{aligned}&\left[\left(k_z^2+\kappa_{mn\lambda}^2\right)-\tfrac{\omega^2}{c^2}\right]C_{mn}^\lambda\left(k_z,\omega\right)-\\&-\tfrac{\omega^2}{c^2}\tfrac{1}{2\pi}\sum_{m'n'\lambda'}\int\vec{Y}_{mn}^{\lambda *}\left(\vec{r}_\perp,k_z\right)\hat{\chi}\left(\vec{r}\right)\vec{Y}_{m'n'}^{\lambda'}\left(\vec{r}_\perp,k_z'\right)e^{-i\left(k_z-k_z'\right)z}\,d^2r_\perp C_{m'n'}^{\lambda'}\left(k_z',\omega\right)dk_z'dz=\\&=-\tfrac{\omega_L^2\left(k_{mn}^2c^2-\omega^2\right)}{\gamma c^4\left(\omega-\vec{k}_{mn}\vec{u}\right)^2}\left\{\tfrac{1}{2\pi}\left|\int\vec{u}\,\vec{Y}_{mn}^\lambda\left(\vec{k}_\perp,k_z\right)d^2k_\perp\right|^2\right\}C_{mn}^\lambda\left(k_z,\omega\right),\end{aligned} \tag{11}$$

where $\vec{Y}_{mn}^\lambda\left(\vec{k}_\perp,k_z\right)=\int e^{-i\vec{k}_\perp\vec{r}_\perp}\vec{Y}_{mn}^\lambda\left(\vec{r}_\perp,k_z\right)d^2r_\perp$.

Note that within the limit where the transverse dimensions of a photonic crystal tend to infinity, the expression between the braces takes the form $\left(\vec{e}\vec{u}\right)^2$, where $\vec{e}$ is the unit polarization vector of the wave emitted by the beam [18,19].

Now let us consider the integrals on the left-hand side of equation (11). Note that according to [15,16,17], the eigenfunctions $\vec{Y}_{mn}^\lambda\left(\vec{r}_\perp,k_z\right)$ of a rectangular waveguide include the combinations of sines and cosines of the form

$sin\frac{\pi m}{a}x, cos\frac{\pi m}{a}x$ $(sin\frac{\pi n}{b}y, cos\frac{\pi n}{b}y)$, i.e., in fact, the combinations $e^{i\frac{\pi m}{a}x}, e^{i\frac{\pi n}{b}y}$. Hence, the left-hand side of the equation includes the integrals of the type

$$I = \int e^{-i\frac{\pi m}{a}x} \sum_i \hat{\chi}_{cell}\left(x - x_i, y - y_i, z - z_i\right) e^{i\frac{\pi m'}{a}x} dx.$$

The substitution of variables $x - x_i = \eta$ gives the sums of the form

$$S_x = \sum_i e^{-i\frac{\pi}{a}(m-m')x_i}$$

where $x_i = d_x f_1$, where d_x is the period of the photonic crystal along the x-axis, $f_1 = 1, 2, ...N_x$, where N_x is the number of cells along the x-axis.

The above-mentioned sum

$$S_x = \sum_i e^{-i\frac{\pi}{a}(m-m')x_i} = e^{i\frac{\pi}{2a}(m-m')(N_x-1)d_x} \frac{sin\frac{\pi(m-m')d_x N_x}{2a}}{sin\frac{\pi(m-m')d_x}{2a}}. \tag{12}$$

If $m - m' = 0$, then $S_x = N_x$.

Let us now discuss what this sum is equal to when $m - m' = 1$. In the numerator $d_x N_x = a$, hence, the nominator is equal to 1 ($sin\frac{\pi}{2} = 1$), while in the denominator $sin\frac{\pi d_x}{2a} \approx \frac{\pi}{2N_x}$. As a result, the relation $\frac{S_x(m-m'=1)}{S_x(m-m'=0)} = \frac{2}{\pi} \approx 0.6$.

With growing difference $m-m'$, the contribution to the sum of the next terms diminishes until the following equality is fulfilled

$$\frac{\pi\left(m - m'\right) d_x}{2a} = \pi \mathrm{P}, \tag{13}$$

where $\mathrm{P} = \pm 1, \pm 2...$ In this case the sum $S_x = N_x$.

The similar reasoning is valid for summation along the y-axis.

It follows from the aforesaid that if the equalities like (12), (13) are fulfilled, that is, the equalities $k_{xm} - k'_{xm'} = \tau_x$ are fulfilled (i.e., $k'_{xm'} = k_{xm} - \tau_x$), where $\tau_x = \frac{2\pi}{d_x}F$ is the x-component of the reciprocal lattice vector of the photonic crystal, $F = 0, \pm 1, \pm 2...$ and $k_{yn} - k'_{yn'} = \tau_y$ (i.., $k'_{yn'} = k_{yn} - \tau_y$), where $\tau_y = \frac{2\pi}{d_y}F'$ is the y-component of the reciprocal lattice vector of the photonic crystal, $F' = 0, \pm 1, \pm 2...$), then the major contribution to the sums comes from the amplitudes $C^{\lambda'}_{m'n'}\left(k'_z, \omega\right) \equiv C^{\lambda'}\left(\vec{k}_{\perp mn} - \vec{\tau}_{\perp}, k_z - \tau_z, \omega\right) = C^{\lambda'}\left(\vec{k}_{mn} - \vec{\tau}, \omega\right)$.

In describing the system we shall further consider only those modes that satisfy the equalities of the type (12), (13). As stated above, the contribution of other modes is suppressed.

As a result, one can rewrite the system of equations (11) as

$$\begin{array}{l}\left(\vec{k}_{mn}^{2}-\frac{\omega^{2}}{c^{2}}\right)C^{\lambda}\left(\vec{k}_{mn},\omega\right)-\frac{\omega^{2}}{c^{2}}\sum\limits_{\lambda'\tau}\chi_{mn}^{\lambda\lambda'}(\vec{\tau})C^{\lambda'}\left(\vec{k}_{mn}-\vec{\tau},\omega\right)=\\ -\frac{\omega_{L}^{2}\left(k_{mn}^{2}c^{2}-\omega^{2}\right)}{\gamma c^{4}\left(\omega-\vec{k}_{mn}\vec{u}\right)^{2}}\left\{\frac{1}{2\pi}\left|\int\vec{u}\,\vec{Y}_{mn}^{\lambda}\left(\vec{k}_{\perp},k_{z}\right)d^{2}k_{\perp}\right|^{2}\right\}C^{\lambda}\left(\vec{k}_{mn},\omega\right),\end{array}\tag{14}$$

i.e.,

$$\begin{array}{l}\left(\vec{k}_{mn}^{2}-\frac{\omega^{2}}{c^{2}}\left(1+\chi_{mn}^{\lambda\lambda}(0)-\frac{\omega_{L}^{2}\left(k_{mn}^{2}c^{2}-\omega^{2}\right)}{\omega^{2}\gamma c^{2}\left(\omega-\vec{k}_{mn}\vec{u}\right)^{2}}\left\{\frac{1}{2\pi}\left|\int\vec{u}\,\vec{Y}_{mn}^{\lambda}\left(\vec{k}_{\perp},k_{z}\right)d^{2}k_{\perp}\right|^{2}\right\}\right)\right)C^{\lambda}\left(\vec{k}_{mn},\omega\right)\\ -\frac{\omega^{2}}{c^{2}}\sum\limits_{\lambda'\tau}\chi_{mn}^{\lambda\lambda'}(\vec{\tau})C^{\lambda'}\left(\vec{k}-\vec{\tau},\omega\right)=0\end{array}\tag{15}$$

where $\chi_{mn}^{\lambda\lambda'}(\tau)=\frac{1}{d_z}\int\vec{Y}_{mn}^{\lambda *}(\vec{r}_{\perp},k_z)\,\hat{\chi}(\vec{r}_{\perp},\tau_z)\vec{Y}_{m'n'}^{\lambda'}(\vec{r}_{\perp},k_z-\tau_z)\,d^2r_{\perp}$, $\hat{\chi}(\vec{r}_{\perp},\tau_z)=\sum\limits_{x_i,y_i}\int\hat{\chi}_{cell}(x-x_i,y-y_i,\zeta)\,e^{-i\tau_z\zeta}d\zeta$, m' and n' are found by the conditions like (13), ω_L is the Langmuir frequency, $\omega_L^2=\frac{4\pi e^2\rho_0}{m}$.

This system of equations coincides in form with that describing the instability of a beam passing through an infinite crystal [18,19]. The difference between them is that the coefficients appearing in these equations are defined differently and that in the case of an infinite crystal, the wave vectors $\vec{k}_{mn}$ have a continuous spectrum of eigenvalues rather than a discrete one.

These equations enable one to define the dependence $k(\omega)$, thus defining the expressions for the waves propagating in the crystal. By matching the incident wave packet and the set of waves propagating inside the photonic crystal using the boundary conditions, one can obtain the explicit expression describing the solution of the considered equations.

The result obtained is formally analogous to that given in [20].

According to (15), the expression between the square brackets acts as the dielectric permittivity ε of the crystal under the conditions when diffraction can be neglected:

$$\varepsilon_0=n^2=1+\chi_{mn}^{\lambda\lambda}(0)-\frac{\omega_L^2\left(k_{mn}^2c^2-\omega^2\right)}{\omega^2\gamma c^2\left(\omega-\vec{k}_{mn}\vec{u}\right)^2}\left\{\frac{1}{2\pi}\left|\int\vec{u}\,\vec{Y}_{mn}^{\lambda}\left(\vec{k}_{\perp},k_z\right)d^2k_{\perp}\right|^2\right\},$$

n is the refractive index.

As is seen, in this case the contribution to the refractive index comes not only from the scattering of waves by the unit cell of the crystal lattice, but also from the scattering of waves by the beam electrons (the term proportional to ω_L^2): the photonic crystal penetrated by a beam of electrons is a medium that can be described by a ceratin refractive index n (or the dielectric permittivity ε_0).

According to (15), the beam contribution increases when $\omega \to \vec{k}\vec{u}$.

Since this system of equations is homogeneous, its solvability condition is the vanishing of the system determinant.

In the beginning, let us assume that the diffraction conditions are not fulfilled. Then the amplitudes of diffracted waves are small. In this case the sum over τ can be dropped, and the conditions for the occurrence of the wave in the system is obtained by the requirement that the expression between the square brackets equal zero.

This expression can be written in the form (the velocity $\vec{u}||oz$)

$$(\omega - k_z u)^2 \left(k_{mn}^2 - \frac{\omega^2}{c^2} n_0^2\right) = -\frac{\omega_L^2 \left(k_{mn}^2 c^2 - \omega^2\right)}{\gamma c^4} \left\{ \frac{1}{2\pi} \left| \int \vec{u}\, \vec{Y}_{mn}^{\lambda} \left(\vec{k}_\perp, k_z\right) d^2 k_\perp \right|^2 \right\},$$

where n_0 is the refractive index of the photonic crystal in the absence of the beam $\varepsilon_0 = n_0^2 = 1 + \chi_{mn}^{\lambda\lambda}(0)$,

i.e.,

$$\left(k_z^2 - \left(\frac{\omega^2}{c^2} n_0^2 - \kappa_{mn}^2\right)\right) (\omega - k_z u)^2 = -\frac{\omega_L^2 \left(k_{mn}^2 c^2 - \omega^2\right)}{\gamma c^4} \left\{ \frac{1}{2\pi} \left| \int \vec{u}\, \vec{Y}_{mn}^{\lambda} \left(\vec{k}_\perp, k_z\right) d^2 k_\perp \right|^2 \right\} \quad (16)$$

Since the nonlinearity is insignificant, let us consider as the zero approximation the spectrum of the waves of equation (16) with zero right-hand side.

Let us concern with the case when $\omega - k_z u \to 0$ (i.e., the Cherenkov radiation condition can be fulfilled) and $\left(k_z^2 - \left(\frac{\omega^2}{c^2} n_0^2 - \kappa_{mn}^2\right)\right) \to 0$, i.e, the electromagnetic wave can propagate in a photonic crystal without the beam. With zero right-hand side the equation reads

$$\left(k_z^2 - \left(\frac{\omega^2}{c^2} n_0^2 - \kappa_{mn}^2\right)\right) = 0, \quad (\omega - k_z u) = 0 \quad (17)$$

As a consequence, in this case the roots of the equation are

$$k_{1z} = \frac{\omega}{c}\sqrt{n_0^2 - \frac{\kappa_{mn}^2 c^2}{\omega^2}}, \quad k'_{1z} = -k_{1z}, \quad k_{2z} = \frac{\omega}{u}. \tag{18}$$

Since $k_{2z} = \frac{\omega}{u} > 0$ in view of the Cherenkov condition, we are concerned with the propagation of the wave with $k_{1z} > 0$ in the photonic crystal. In this case in the equation for k_z, one can take $(k_z - k_{1z})(k_z + k_{1z}) \approx 2k_{1z}(k_z - k_{1z})$ and rewrite equation (16) as follows:

$$(k_z - k_{1z})(k_z - k_{2z})^2 = -\frac{\omega_L^2 \omega^2 (n_0^2 - 1)}{2k_{1z} u^2 \gamma c^4} \left\{ \frac{1}{2\pi} \left| \int \vec{u}\, \vec{Y}_{mn}^{\lambda}\left(\vec{k}_\perp, k_z\right) d^2 k_\perp \right|^2 \right\} \tag{19}$$

i.e.,

$$(k_z - k_{1z})(k_z - k_{2z})^2 = -A \tag{20}$$

where A is real and $A > 0$ (as for the occurrence of the Cherenkov effect, it is necessary that $n_0^2 > 1$). We have obtained the cubic equation for k_z. Let us consider the case when the roots k_{1z} and k_{2z} coincide $k_{1z} = k_{2z}$. It is possible when the particle velocity satisfies the condition

$$u = \frac{c}{\sqrt{n_0^2 - \frac{\kappa_{mn}^2 c^2}{\omega^2}}}. \tag{21}$$

Introduction of $\xi = k - k_{1z}$ gives for $k_{1z} = k_{2z}$

$$\xi^3 = -A. \tag{22}$$

The solution of equation (22) gives three roots $\xi_1 = -\sqrt[3]{A}$, $\xi_{2,3} = \frac{1}{2}\left(1 \pm i\sqrt{3}\right)\sqrt[3]{A}$.

As a consequence, the state corresponding to the root $\xi_2 = \frac{1}{2}\left(1 + i\sqrt{3}\right)\sqrt[3]{A}$ grows with growing z, which indicates the presence of instability in a beam [21]. In this case $Im\, k_z = Im\, \xi_2 \sim \sqrt[3]{\rho}$.

Note here that the photonic crystal built from metallic threads has the refractive index $n_0 < 1$ for a wave with the electric polarizability parallel to the threads, i.e., in this case the Cherenkov instability of the beam does not exist [9] (but if the electric vector of the wave is orthogonal to the metallic threads, the refractive index is $n_0 > 1$, so for such a wave the Cherenkov instability exists [22]).

It should be pointed out, however, that, unlike an infinite photonic crystal, the field in the crystal placed into the waveguide has a mode character, and so the presence of κ_{mn}^2 in the denominator of equation (21) results in reduction of the radicand in (21) to the magnitude smaller than unity even when $n_0^2 > 1$. Hence, $u > c$, which is impossible. Consequently, the radiative instability of the above type in the waveguide can arise under the condition $n_0^2 - \frac{\kappa_{mn}^2 c^2}{\omega^2} > 1$ rather than $n_0^2 > 1$.

Suppose now that in the photonic crystal the conditions can be realized under which the wave amplitude $C_{mn}\left(\vec{k}_{mn} + \vec{\tau}\right)$ is comparable with the amplitude $C_{mn}\left(\vec{k}_{mn}\right)$. By analogy with the standard diffraction theory for an infinite crystal [23,24], in the case under consideration, when $\chi << 1$, it is sufficient that only the equations for these amplitudes remain in (15).

To be specific, let us further consider a photonic crystal formed by parallel threads. Also assume that they are parallel to the waveguide boundary (y, z).

Analysis of diffraction of a λ-type wave with the electric vector in the plane (y, z) (a TM-wave) gives

$$\left[k_{mn}^2 - \frac{\omega^2}{c^2}\varepsilon\right] C^\lambda\left(\vec{k}_{mn}, \omega\right) - \frac{\omega^2}{c^2}\chi_{mn}^{\lambda\lambda}\left(-\vec{\tau}\right) C^\lambda\left(\vec{k}_{mn} + \vec{\tau}, \omega\right) = 0 \tag{23}$$

$$\left[\left(\vec{k}_{mn} + \vec{\tau}\right) - \frac{\omega^2}{c^2}\varepsilon_0\right] C^\lambda\left(\vec{k}_{mn} + \vec{\tau}, \omega\right) - \frac{\omega^2}{c^2}\chi_{mn}^{\lambda\lambda}\left(\vec{\tau}\right) C^\lambda\left(\vec{k}_{mn}, \omega\right) = 0.$$

Since the term containing $\left(\omega - \left(\vec{k} + \vec{\tau}\right)\vec{u}\right)^{-1}$ is small when $\left(\omega - \vec{k}\vec{u}\right)$ vanishes, in the second equation it is dropped.

The dispersion equation defining the relation between k_z and ω is obtained by equating to zero the determinant of the system (23) and has a form:

$$\begin{gathered}\left[\left(k_{mn}^2 - \tfrac{\omega^2}{c^2}\varepsilon_0\right)\left(\left(\vec{k}_{mn} + \vec{\tau}\right)^2 - \tfrac{\omega^2}{c^2}\varepsilon_0\right) - \tfrac{\omega^4}{c^4}\chi_\tau\chi_{-\tau}\right]\left(\omega - k_z u\right)^2 = \\ -\tfrac{\omega_L^2}{\gamma c^4}\left\{\tfrac{1}{2\pi}\left|\int \vec{u}\,\vec{Y}_{mn}^\lambda\left(\vec{k}_\perp, k_z\right) d^2k_\perp\right|^2\right\}\left(k_{mn}^2 c^2 - \omega^2\right)\left(\left(\vec{k}_{mn} + \vec{\tau}\right)^2 - \tfrac{\omega^2}{c^2}\varepsilon_0\right).\end{gathered} \tag{24}$$

Because the right-hand side of the equation is small, one can again seek the solution near the points where the right-hand side is zero that corresponds the condition of occurrence of the Cherenkov radiation and excitation of the wave which can propagate in the waveguide:

$$\begin{gathered}\left(k_z^2 - \left(\tfrac{\omega^2}{c^2}\varepsilon_0 - \kappa_{mn}^2\right)\right)\left(\left(k_z + \tau\right)^2 - \left(\tfrac{\omega^2}{c^2}\varepsilon_0 - \left(\vec{\kappa}_{mn} + \vec{\tau}_\perp\right)^2\right)\right) - \tfrac{\omega^4}{c^4}\chi_\tau\chi_{-\tau} = 0 \\ \left(k_z - \tfrac{\omega}{u}\right)^2 = 0\end{gathered} \tag{25}$$

The roots of equations are sought near the conditions $k_{mn}^2 \approx \left(\vec{k}_{mn} + \vec{\tau}\right)$,

$$k_z = k_{z0} + \xi, \quad k_z^2 = k_{z0}^2 + 2k_{z0}\xi + \xi^2, \quad k_{z0}^2 = \frac{\omega^2}{c^2}\varepsilon_0 - \kappa_{mn}^2; \quad k_{z0} = \frac{\omega}{c}\sqrt{\varepsilon_0 - \frac{\kappa_{mn}^2 c^2}{\omega^2}} \tag{26}$$

$$\left(k_z + \tau_z\right)^2 = \left[\left(k_{0z} + \tau_z\right) + \xi\right]^2 = \left(k_{0z} + \tau_z\right)^2 + 2\left(k_{0z} + \tau_z\right)\xi + \xi^2$$

Hence,

$$\begin{gathered}\left(k_{0z} + \tau_z\right)^2 + \left(\vec{\kappa}_{mn} + \vec{\tau}_\perp\right)^2 + 2\left(k_{0z} + \tau_z\right) + 2\left(k_{0z} + \tau_z\right)\xi + \xi^2 = \\ \left(\vec{k}_{mn} + \vec{\tau}\right)^2 + 2\left(k_{0z} + \tau_z\right)\xi + \xi^2 = k_{0mn}^2 + 2\vec{k}_{0mn}\vec{\tau} + \tau^2 + 2\left(k_{0z} + \tau_z\right)\xi + \xi^2.\end{gathered} \tag{27}$$

And one can get

$$\begin{gathered}2k_{0z}\xi\left(2\left(k_{0z} + \tau_z\right)\xi + \left(2\vec{k}_{0mn}\vec{\tau} + \tau^2\right)\right) - \frac{\omega^4}{c^4}\chi_\tau\chi_{-\tau} = 0 \\ 4k_{0z}\left(k_{0z} + \tau_z\right)\xi^2 + 2k_{0z}\left(2\vec{k}_{0mn}\vec{\tau} + \tau^2\right)\xi - \frac{\omega^4}{c^4}\chi_\tau\chi_{-\tau} = 0 \\ \xi^2 + \frac{\left(2\vec{k}_{0mn}\vec{\tau} + \tau^2\right)}{\left(k_{0z} + \tau_z\right)}\xi - \frac{\omega^4}{c^4}\frac{\chi_\tau\chi_{-\tau}}{4k_{0z}\left(k_{0z} + \tau_z\right)} = 0 \\ \xi_{1,2} = -\frac{\left(2\vec{k}_0\vec{\tau} + \tau^2\right)}{4\left(k_{0z} + \tau_z\right)} \pm \sqrt{\left(\frac{2\vec{k}_0\vec{\tau} + \tau^2}{4\left(k_{0z} + \tau_z\right)}\right)^2 + \frac{\omega^4}{c^4}\frac{\chi_\tau\chi_{-\tau}}{4k_{0z}\left(k_{0z} + \tau_z\right)}}\end{gathered} \tag{28}$$

If $\left(k_{0z} + \tau_z\right) = -\left|k_{0z} + \tau_z\right|$, the root can cross the zero point. At the same time, the second equation should hold.

$$\omega - k_z u = \omega - k_{0z}u - \xi u = 0.$$

Consequently,

$$\xi = \frac{\omega - k_{0z}u}{u} = \frac{\omega}{u} - k_{0z} = \frac{\omega}{u} - \frac{\omega}{c}\sqrt{\varepsilon_0 - \frac{\kappa_{mn}^2 c^2}{\omega^2}}.$$

If $\varepsilon_0 < 1$, then $\xi = \frac{\omega}{u}\left(1 - \beta\sqrt{\varepsilon_0 - \frac{\kappa_{mn}^2 c^2}{\omega^2}}\right) > 0$, $\xi = \frac{\omega}{u} - k_{0z}$

Let the roots ξ_1 and ξ_2 coincide ($\xi_1 = \xi_2$). This is possible at point

$$\frac{2\vec{k}_0\vec{\tau} + \tau^2}{4\left(k_{0z} + \tau_z\right)} = \pm\frac{\omega^2}{c^2}\frac{\sqrt{\chi_\tau\chi_{-\tau}}}{\sqrt{4k_{0z}\left|k_{0z} + \tau_z\right|}},$$

here $k_{0z}+\tau_z<0$.

The roots coincide when the following equality is fulfilled

$$\frac{\omega}{u}-k_{0z}=\mp\frac{\omega^2}{c^2}\frac{\sqrt{\chi_\tau\chi_{-\tau}}}{\sqrt{4k_{0z}\left|k_{0z}+\tau_z\right|}},$$

i.e.,

$$\frac{\omega}{u}=k_{0z}\mp\frac{\omega^2}{c^2}\frac{\sqrt{\chi_\tau\chi_{-\tau}}}{\sqrt{4k_{0z}\left|k_{0z}+\tau_z\right|}}\quad\text{and}\quad k_{0z}=\frac{\omega}{c}\sqrt{\varepsilon_0-\frac{\kappa_{mn}^2c^2}{\omega^2}}$$

Let $\varepsilon_0<1$, then $\frac{\omega}{u}>k_{0z}$ (since $u<c$), the situation for the solution $\frac{\omega}{u}=k_{0z}-\frac{\omega^2}{c^2}\frac{\sqrt{\chi_\tau\chi_{-\tau}}}{\sqrt{4k_{0z}\left|k_{0z}+\tau_z\right|}}$ gets complicated and the Vavilov-Cherenkov condition is not fulfilled.

Now let us consider the solution $\frac{\omega}{u}=k_{0z}+\frac{\omega^2}{c^2}\frac{\sqrt{\chi_\tau\chi_{-\tau}}}{\sqrt{4k_{0z}\left|k_{0z}+\tau_z\right|}}$. At $\tau_z<0$ the difference $k_{0z}+\tau_z$ can be reduced so that the sum on the right would appear to become equal to $\frac{\omega}{u}$, and so one could obtain 4 coinciding roots.

Interestingly enough, for backward diffraction, which is a typical case of frequently used one-dimensional generators with a corrugated metal waveguide (the traveling-wave tube, the backward-wave tube), such a coincidence of roots is impossible.

Indeed, let the roots ξ_1 and ξ_2 coincide. In this case for the backward Bragg diffraction $|\tau_z|\approx 2k_{0z}$, $\tau_z<0$. Then by substituting the expressions for $k_{0z}=\frac{\omega}{c}\sqrt{\varepsilon_0-\frac{\kappa_{mn}^2c^2}{\omega^2}}$ and $\varepsilon_0=n_0^2=1+\chi_{mn}^{\lambda\lambda}(0)$ and retaining the first-order infinitesimal terms, the relation $\frac{\omega}{u}\approx k_{0z}+\frac{\omega^2}{c^2}\frac{|\chi_\tau|}{2k_{0z}}$ can be reduced to the form $\frac{\omega}{u}\approx\frac{\omega}{c}\left(1-\frac{\left|\chi_{mn}^{\lambda\lambda}(0)\right|}{2}-\frac{\kappa_{mn}^2c^2}{2\omega^2}+\frac{\omega}{c}\frac{|\chi_\tau|}{2}\right)<\frac{\omega}{u}$, i.e., the equality does not hold and the four-fold degeneracy is impossible. Only ordinary three-fold degeneration is possible.

However, if $\varepsilon_0>1$ and is appreciably large, then in a one-dimensional case, the four-fold degeneracy of roots is also possible in a finite photonic crystal[1].

Thus, the left-hand side of equation (24) has four roots ξ_1, ξ_2, and a double

[1] The authors are grateful to K. Batrakov, who drew our attention to the fact that for an infinite crystal with $\varepsilon_0>1$, the intersection of roots is possible in a one-dimensional case.

degenerated root ξ_3. Hence, equation (24) can be written as follows:

$$(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)^2 = B.$$

If the roots coincide ($\xi_1 = \xi_2 = \xi_3$), one obtains $(\xi - \xi_1)^4 = B$,, i.e., $\xi - \xi_1 = \sqrt[4]{B}$.

The fourth root of B has imaginary solutions depending on the beam density as $Imk_z \sim \rho_0^{1/4}$ (the parameter $B \sim \omega_L^2$, i.e., $B \sim \rho_0$, see the right-hand side of (24)). This increment is larger than the one we obtained for the case of the three-fold degeneracy.

The analysis shows that with increasing number of diffracted waves, the law established in [6,7,8] is valid: the instability increment appears to be proportional to $\rho^{\frac{1}{s+3}}$, where s is the number of waves emerging through diffraction. As a result, the abrupt decrease in the threshold generation current also remains in this case (the threshold generation current $j_{th} \sim \frac{1}{(kL)^3 (k\chi_\tau L)^{2s}}$, where L is the length of the interaction area).

It is interesting that according to [12], for a photonic crystal made from metallic threads, the coefficients $\chi(\tau)$, defining the threshold current and the growth of the beam instability, are practically independent on τ up to the terahertz range of frequencies because the diameter of the thread can easily be made much smaller than the wavelength. That is why photonic crystals with the period of about 1 mm can be used for lasing in terahertz range at high harmonics (for example, photonic crystal with 3 mm period provides the frequency of the tenth harmonic of about 1 terahertz (λ=300 micron).

The analysis of laser generation in VFEL with a photonic crystal when the beam moves in an undulator (electromagnetic wave) located in a finite crystal, made similarly to the above analysis, shows that in this case the dispersion equation and the law of instability also have the same form as in the case of an infinite crystal. The procedure for going from the dispersion equations describing instability in the infinite case (15) is similar to that discussed earlier in this paper. It consists in replacing the continuous $\vec{k}$ by the quantified values of $\vec{k}_{mn}$ and redefining the coefficients appearing in equations like (15).

It is important to emphasize the general character of the rules found in this paper for obtaining dispersion equations that describe the radiative instability of the electron beam in a finite photonic crystal. In particular, they are valid for describing the processes of instability of an electromagnetic wave in finite nonlinear photonic crystals.

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